Notes on Deontic Logic*

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Deontic logic is the formal study of the normative concepts of obligation, permission, and prohibition. These concepts and their logical relationships to one another are distinguished from value concepts such as goodness and badness (or evil), as well as from such agent-based concepts as act, choice, decision, desire, freedom, and will.

A deontic logic is not itself an ethical theory that tells us what in fact is permitted, obligatory, or forbidden, but it is, or should be, part of such a theory. A complete ethical or moral theory would encompass the logic of all of these different concepts, and not just the normative ones. But just as there can be different ethical theories, so too there can be different deontic logics. The study of the different deontic logics, in other words, is part of what is more properly called meta-ethics.

The normative concepts of obligation and permission are similar in many respects to the modal concepts of necessity and possibility. In fact the normative concepts are also modal concepts themselves. A deontic logic, in other words, is a modal logic where instead of the unary formula (sentential) operators for the alethic modalities necessity and possibility, namely $\Box$ and $\Diamond$, we have unary operators for the deontological modalities of obligation and permission, namely $\mathcal{O}$ and $\mathcal{P}$.

Alethic logic deals with what is the case (truth) or not the case (falsehood), and alethic modal logic deals with what is necessarily or possibly the case as well as with what is the case or not the case. Deontic modal logic deals with what is obligatory, or permitted, or forbidden to be the case as well as with what is or is not the case.

We will understand formulas, i.e., the sentential (propositional) forms of deontic logic, to stand for propositions, or states of affairs, or for actions in the case of a background agency and action theory, except that this usually requires a development of the logic of actions as well, which we will not go into here.

1 Logical Grammar

The logical grammar for sentential (propositional) deontic logic consists of the unary formula operator for negation, $\neg$, and the binary formula operator for the (material) conditional $\rightarrow$. The (material) biconditional $\leftrightarrow$, conjunction $\wedge$, and

*These notes are based on a course I gave on modal logic in the late 1960s at the State University of California at San Francisco.

disjunction $\vee$ are binary operators that are assumed to be defined in the usual way in terms of negation and the (material) conditional operators.

We will use the lower case Greek letters φ, ψ, χ as metalanguage variables that range over the formulas of our object language.

We read the deontic operators $\mathcal{O}$ and $\mathcal{P}$ as follows:

$\mathcal{O}\varphi$: It is obligatory that φ (or one is obligated to do φ).
$\mathcal{P}\varphi$:It is permitted that φ (or one is permitted to do φ).

With $\neg$ as the sign for negation, we give $\neg\mathcal{P}\varphi$ a special reading as well:

$\neg\mathcal{P}\varphi$: It is forbidden (or prohibited) that φ (or one is forbidden to do φ),

which we can symbolize more simply as $\mathcal{F}\varphi$, i.e.,

$$\mathcal{F}\varphi \leftrightarrow \neg\mathcal{P}\varphi.$$

As we will see, because $\neg\mathcal{P}\varphi$ is equivalent to $\mathcal{O}\neg\varphi$, we also have:

$$\mathcal{F}\varphi \leftrightarrow \mathcal{O}\neg\varphi.$$

We will also assume that the permission operator is defined in terms of the obligation operator as follows:

$$\mathcal{P}\varphi \leftrightarrow \neg\mathcal{O}\neg\varphi.$$

That is, for φ to be permitted means that it is not the case that $\neg\varphi$ is obligatory. This means that we take only $\mathcal{O}$, $\neg$, and $\rightarrow$ as primitive, with $\mathcal{P}$, $\mathcal{F}$, $\leftrightarrow$, $\wedge$, and $\vee$ assumed to be defined as abbreviatory signs.

Note: A unary formula operator applies to one formula—i.e., a sentence form—and results in a formula. A binary formula operator, such as $\rightarrow$, applies to two formulas and results in a formula.

Where φ and ψ are arbitrary formulas (sentence forms), these operators are read as follows:

$\neg\varphi$: It is not the case that φ.
$(\varphi \rightarrow \psi)$: If φ, then ψ.
$(\varphi \leftrightarrow \psi)$: φ if, and only if, ψ.
$(\varphi \wedge \psi)$: φ and ψ.
$(\varphi \vee \psi)$: φ or ψ.

We will assume a potentially infinite list of sentential (propositional) variables $p, q, r, p_1, q_1 ..., p_n, ...$ for all integers n. We inductively define the formulas of deontic logic as the smallest set of formulas containing the sentential variables and such that $\neg\varphi$, $\mathcal{O}\varphi$, and $(\varphi \rightarrow \psi)$ belong to the set whenever φ, ψ belong to the set.

2 Systems of Deontic Logic

There are a number of possible choices that one can make in regard to what deontic logic to adopt. Much will depend on various meta-ethical considerations. What we will do here is consider the principle theses of different deontic logics and leave open the choice of which system to adopt. Because of the similarity of the concepts of permission and obligation with the concepts of possibility and necessity, our methodology will be to note each of the principle theses of alethic modal logic and then consider whether or not the deontic counterparts of these theses are acceptable principles of deontic logic.

We will assume that all tautologous formulas (as based on the formulas of sentential deontic logic) are valid in all deontic logics, and hence, because tautologies are decidable, we will assume all tautologies to be derivable from a single axiom schema of every system of deontic logic considered here.

We will also assume that all of the systems of sentential deontic logic have the same two primitive inference rules, namely, *modus ponens*, (MP), and the rule, (O), that what is provable in deontic logic is obligatory:

Modus ponens (MP): If $\vdash \varphi$ and $\vdash (\varphi \rightarrow \psi)$, then $\vdash \psi$.

Obligatory (O): If $\vdash \varphi$, the $\vdash \mathcal{O}\varphi$.

The symbol $\vdash$ used above is read as 'is provable'. For a particular logistic system Σ , we read $\vdash_\Sigma$ as 'is provable in Σ. If $\Gamma \cup \{\varphi\}$ is a set of formulas, then we take $\Gamma \vdash \varphi$ to mean that Γ yields φ in Σ, i.e., that φ is derivable from a (finite) number of formulas in Γ as premises.

Because the axioms and rules of all of the systems considered here will be stated schematically, the rule of uniform substitution (US) of a formula for a propositional variable is derivable in each system Σ:

Uniform substitution (US): If $\vdash_\Sigma \varphi$, then $\vdash_\Sigma \varphi(p_n/\psi)$.

The systems of alethic modal logic that we mention here can be ordered in terms of two lines of the subsystem relation:

$$Kr \sqsubseteq M \sqsubseteq S4 \sqsubseteq S4.2 \sqsubseteq S4.3 \sqsubseteq S5,$$
$$Kr \sqsubseteq M \sqsubseteq Br \sqsubseteq S5.$$

The deontic logics that are counterparts of these alethic modal logics are ordered similarly, with the inclusion of one further system, sometimes known as standard deontic logic, which we will call D:

$$DKr \sqsubseteq D \sqsubseteq DM \sqsubseteq DS4 \sqsubseteq DS4.2 \sqsubseteq DS4.3 \sqsubseteq DS5,$$
$$DKr \sqsubseteq D \sqsubseteq DM \sqsubseteq DBr \sqsubseteq DS5.$$

Finally, because all of the systems considered here will include the principles of DKr, the interchange rule that allows provable formulas to be interchanged in more complex formulas is also derivable in deontic logic.

3 The Deontic Logic *DKr*

The minimal system of alethic modal logic is the system Kr, which, in addition to the axioms for tautologies, consists of a single axiom for the distribution of $\Box$ over a conditional, i.e., the thesis that if a conditional is necessary, then the antecedent is nessary only if the consequent is as well. The deontic counterpart of this thesis, namely that if a conditional is obligatory then the antecedent is obligatory only if the consequent is as well, is clearly an acceptable thesis of deontic logic. Accordingly, the axioms of the minimal system DKr of deontic logic can be described as follows:

Axiom1: $\vdash_{DKr} \varphi$ if φ is a tautology,

Axiom 2: $\vdash_{DKr} \mathcal{O}(\varphi \to \psi) \to (\mathcal{O}\varphi \to \mathcal{O}\psi)$.

Axiom 2, the principle deontic thesis of DKr, and the rule (O) lead to what some consider another oddity if not a paradox. This is the fact $\vdash_{DKr} \mathcal{O}\varphi \to \mathcal{O}(\varphi \vee \psi)$. In other words because $\varphi \to \varphi \vee \psi$ is a tautology, then, by the rule (O), $\vdash_{DKr} \mathcal{O}(\varphi \to \varphi \vee \psi)$, and therefore by axiom 2 and modus ponens, $\vdash_{DKr} \mathcal{O}\varphi \to \mathcal{O}(\varphi \vee \psi)$. (This is sometimes called **Ross's paradox**, after the Danish philosopher of law, Alf Ross.)

This theorem is said to be odd, even if not actually paradoxical, because, e.g., it then follows that if it ought to be that we love our neighbor, then it ought to be that we either love our neighbor or hate our neighbor. Hence, given that we ought to love our neighbors, then it follows that we ought to love our neighbors or hate them.

But is this really odd? Given that a proposition p is true, it then follows that $p \vee (p \wedge \neg p)$ is also true. The truth of p in this case conveys no truth to the contradiction $(p \wedge \neg p)$, even though the disjunction $p \vee (p \wedge \neg p)$ is also true. Similarly, the obligation to love our neighbors conveys no obligation to hate our neighbors; and in fact given a prohibition against hating our neighbors, the obligation to hate our neighbors would then be refutable just as $(p \wedge \neg p)$ is refutable.

There is said to be a more serious problem with axiom 2 and the rule (O), however. In particular, they are said to lead to **the Good Samaritan Paradox**, which we can describe as follows:

> If the good Samaritan helps Paul who has been robbed, then Paul has been robbed. (A tautology)
>
> Therefore, It ought to be that (if the good Samaritan helps Paul who has been robbed, then Paul has been robbed). (By the rule (O))
>
> But it ought to be that the good Samaritan helps Paul who has been robbed. (Assumption)
>
> Therefore, it ought to be that Paul has been robbed. (By axiom 2 and modus ponens).
>
> But clearly this conclusion is false; hence we have a contradiction.

We could replace axiom 2 by the weaker rule:

$$\text{If} \vdash \varphi \to \psi, \text{ then} \vdash \mathcal{O}\varphi \to \mathcal{O}\psi.$$

In terms of action theory the rule states:

If doing φ logically implies doing ψ, then
one ought to do φ only if one ought to do ψ.

Unfortunately, the Good Samaritan Paradox is still derivable on the basis of this weaker rule even without axiom 2 or the rule (O).

But there is nothing really left to deontic logic if we give up axiom 2 and the rule (O).

Question: How can we escape the Good Samaritan Paradox?

If we accept axiom 2 and the rule (O), then it is only the assumption (premise 3) that can be challenged. Is it really true that it ought to be that the Good Samaritan helps Paul who has been robbed?

Let us note that logically the sentence 'The Good Samaritan helps Paul who has been robbed' is really a conjunction, namely the conjunction that Paul has been robbed **and** that the Good Samaritan helps him (Paul). The assumption then is really saying:

> *It ought to be that* **(**Paul has been robbed and the Good Samaritan helps him**)**,

which by a theorem of DKr, namely, $\mathcal{O}(\varphi \wedge \psi) \leftrightarrow \mathcal{O}\varphi \wedge \mathcal{O}\psi$, is equivalent to saying:

> *It ought to be that* Paul has been robbed **and** *it ought to be that* the Good Samaritan helps him.

In other words the assumption, premise 3, implicitly assumes the conclusion that we reject. The solution is that the premise is not true after all.

Note that If instead of Paul we referred simply to someone, i.e., if the assumption were replaced by the different statement that it ought to be the case that the Good Samaritan helps someone who has been robbed, then even though it is less obvious that the assumption implicitly assumes that it ought to be that someone is robbed, nevertheless that in fact is what is being assumed and hence again there is no paradox except our assuming what we are rejecting.

Note: Perhaps a more interesting version of the paradox can be formulated in terms the *de re-de dicto* distinction with respect to the deontic operators, i.e., in terms of variable-binding operators reaching into a deontic context. The assumption, for example, might be restated as a *de re* obligation of the Good Samaritan, namely that he ought to help Paul:

> The Good Samaritan is (an x) such that *It ought to be that* [*he* (x) helps Paul who has been robbed].

Evaluating this version of the assumption would take us into quantified deontic logic, which we are not doing here. But we might note that stated this way the premise may depend on the circumstances of the situation. Before the good Samaritan found the person who was robbed, others had passed him by without helping him. Had one of those helped the person who was robbed, there would then be no need, and hence no obligation, for the good Samaritan to help him. Also, what if the good Samaritan has no resources by which to help the person who is robbed? What if he is blind and does not even know that a person who has been robbed is near him. Does the *de re* obligation apply even in these cases?

It is important to note here that it may be obligatory that someone do φ, without it being the case that some particular person has a *de re* obligation to do φ. At a beach by the sea crowded with people where a child is drowning and there is no lifeguard, it is obligatory that someone try to rescue the child; but it does not follow that there is someone who is such that he or she ought to try to rescue the child.

The idea that a *de re* obligation may depend on circumstances in different situations suggests that a conditional binary concept of obligation and similarly of permission may be more appropriate than our monadic concepts. The formula $\mathcal{O}(\varphi/\psi)$, e.g., might be read as 'it is obligatory that φ given the circumstances that ψ'. We will not pursue this suggestion here.

Despite being the minimal system of deontic logic, there are a number of theses provable in DKr that indicate some of the important connections between the standard sentential connectives and the deontic operators.

1. $\vdash_{DKr} \mathcal{P}\varphi \leftrightarrow \neg\mathcal{O}\neg\varphi$,
2. $\vdash_{DKr} \neg\mathcal{O}\varphi \leftrightarrow \mathcal{P}\neg\varphi$,
3. $\vdash_{DKr} \mathcal{O}\varphi \leftrightarrow \neg\mathcal{P}\neg\varphi$,
4. $\vdash_{DKr} \neg\mathcal{P}\varphi \leftrightarrow \mathcal{O}\neg\varphi$.
5. $\vdash_{DKr} \mathcal{O}(\varphi \wedge \psi) \leftrightarrow \mathcal{O}\varphi \wedge \mathcal{O}\psi$,
6. $\vdash_{DKr} \mathcal{P}(\varphi \vee \psi) \leftrightarrow \mathcal{P}\varphi \vee \mathcal{P}\psi$,
7. $\vdash_{DKr} \mathcal{O}(\varphi \to \psi) \to (\mathcal{P}\varphi \to \mathcal{P}\psi)$,
8. $\vdash_{DKr} \mathcal{O}(\varphi \leftrightarrow \psi) \to (\mathcal{O}\varphi \leftrightarrow \mathcal{O}\psi)$,
9. $\vdash_{DKr} \mathcal{O}(\varphi \leftrightarrow \psi) \to (\mathcal{P}\varphi \leftrightarrow \mathcal{P}\psi)$,
10. $\vdash_{DKr} \neg\mathcal{P}\varphi \to \mathcal{O}(\varphi \to \psi)$,
11. $\vdash_{DKr} \mathcal{O}\psi \to \mathcal{O}(\varphi \to \psi)$,
12. $\vdash_{DKr} \neg\mathcal{P}\varphi \leftrightarrow \mathcal{O}(\varphi \to \psi) \wedge \mathcal{O}(\varphi \to \neg\psi)$,
13. $\vdash_{DKr} \mathcal{O}\varphi \vee \mathcal{O}\psi \to \mathcal{O}(\varphi \vee \psi)$,

14. $\vdash_{DKr} \mathcal{P}(\varphi \wedge \psi) \rightarrow \mathcal{P}\varphi \wedge \mathcal{P}\psi$,

15. $\vdash_{DKr} (\mathcal{P}\varphi \rightarrow \mathcal{O}\psi) \rightarrow \mathcal{O}(\varphi \rightarrow \psi)$,

16. $\vdash_{DKr} \mathcal{P}\varphi \wedge \mathcal{O}\psi \rightarrow \mathcal{P}(\varphi \wedge \psi)$,

17. $\vdash_{DKr} \mathcal{P}(\varphi \rightarrow \psi) \leftrightarrow (\mathcal{O}\varphi \rightarrow \mathcal{P}\psi)$,

18. $\vdash_{DKr} \mathcal{F}(\varphi \vee \psi) \leftrightarrow \mathcal{F}\varphi \wedge \mathcal{F}\psi$.

Note: Theorem 6 tells us that permission distributes over a disjunction, i.e., that a disjunctive permission is equivalent to a disjunction of permissions. A disjunctive permission does not imply a conjunction of permissions, however. There is another notion, which is sometimes called "free choice" permission, for which a disjunctive permission does imply (and is equivalent to) a conjunction of permissions. For example, when it is permitted that you vote for candidate A or vote for candidate B, then it is permitted that you vote for candidate A *and* it is permitted that you vote for candidate B.

But because $p \rightarrow p \vee (p \wedge \neg p)$ is a tautology, then, by the rule (O), $\mathcal{O}[p \rightarrow p \vee (p \wedge \neg p)]$ is provable, and therefore, by axiom 2 and theorem 7, $\mathcal{P}p \rightarrow \mathcal{P}(p \vee [p \wedge \neg p])$ is also provable. Accordingly, if we were to allow free-choice permission, we would have $\mathcal{P}p \rightarrow \mathcal{P}(p \wedge \neg p)$ as provable, from which it would follow that nothing is permitted on pain otherwise of obtaining the contradiction $\mathcal{P}(p \wedge \neg p) \wedge \neg\mathcal{P}(p \wedge \neg p)$.

4 The Deontic Logic *D*

The system DKr is of interest because it is minimal in the sense that the alethic modal logic Kr is minimal; in particular both strongly characterize the class of all world systems, i.e., ordered systems of possible worlds. A real difference between deontic and alethic modal logic begins with the deontic counterpart of the alethic modal logic M, the principle thesis of which is $\Box\varphi \rightarrow \varphi$, i.e., that what is necessary is the case.

Of course, the counterpart of this thesis, namely $\mathcal{O}\varphi \rightarrow \varphi$, is not a valid thesis of deontic logic. It is not inconsistent because it could possibly be true (for all instances of φ), but only in an ideal world where whatever is obligatory is in fact the case. Given quantification over propositions or states of affairs, we could take this characterization of an ideal world as a definition:

Definition: A possible world W is *ethically ideal* if, and only if, for all formulas φ, $(\mathcal{O}\varphi \rightarrow \varphi)$ is true in W. Or in terms of actions, a world W is ideal iff every action that ought to be done in W is in fact done in W.

The relevant deontic counterpart of the alethic modal thesis $\Box\varphi \rightarrow \varphi$ is the weaker thesis that *what is obligatory is permitted*: $\mathcal{O}\varphi \rightarrow \mathcal{P}\varphi$, which is equivalent to the thesis that there cannot be conflicting obligations: $\neg(\mathcal{O}\varphi \wedge \mathcal{O}\neg\varphi)$, i.e., one cannot be obligated to both do and not do φ We will refer to the system

that results by adding this principle to those of DKr as the deontic logic D. As we will see, although DKr (and Kr) strongly characterize the class of all world systems, the deontic logic D strongly characterizes the class of all deontic world systems, which we will characterize later.
The axioms of D are the following:

Axiom1: $\vdash_{DM} \varphi$ if φ is a tautology,

Axiom 2: $\vdash_{DM} \mathcal{O}(\varphi \to \psi) \to (\mathcal{O}\varphi \to \mathcal{O}\psi)$.

Axiom 3: $\vdash_{DM} \mathcal{O}\varphi \to \mathcal{P}\varphi$.

5 The Deontic Logic *DM*

Another weaker counterpart of the M thesis $\Box\varphi \to \varphi$ is the claim it ought to be that what ought to be is the case, i.e., $\mathcal{O}(\mathcal{O}\varphi \to \varphi)$. In other words, even though the real world is not ethically ideal, nevertheless it ought to be ideal.

Adding this principle to D gives us the deontic logic DM. The axioms of DM are the then following:

Axiom1: $\vdash_{DM} \varphi$ if φ is a tautology,

Axiom 2: $\vdash_{DM} \mathcal{O}(\varphi \to \psi) \to (\mathcal{O}\varphi \to \mathcal{O}\psi)$.

Axiom 3: $\vdash_{DM} \mathcal{O}\varphi \to \mathcal{P}\varphi$.

Axiom 4: $\vdash_{DM} \mathcal{O}(\mathcal{O}\varphi \to \varphi)$.

Of course, because the axioms and inference rules of DKr are axioms and inference rules of DM, it follows that all of the theorems of DKr are theorems of DM.

The following are theorems that are provable in DM but not in DKr, and as such they indicate new connections between the sentential and deontic operators:

1. $\vdash_{DM} \mathcal{O}(\varphi \to \mathcal{P}\varphi)$.
2. $\vdash_{DM} \mathcal{O}\mathcal{O}\varphi \to \mathcal{O}\varphi$.
3. $\vdash_{DM} (\mathcal{P}\varphi \to \mathcal{P}\psi) \to \mathcal{P}(\varphi \to \psi)$.

In terms of actions, we can read theorem 3 as: if doing φ is permitted only if doing ψ is permitted, then it is permitted that one does φ only if one does ψ.

There is a question about the principle thesis of DM, i.e., the thesis $\mathcal{O}(\mathcal{O}\varphi \to \varphi)$, which, as noted, says in effect that the real world ought to be ideal. The question is: Can there be free will in an ethically ideal world where what ought to be the case is the case, and hence where what is forbidden is not the case? Does not free will require permitting something that is forbidden to be the case, i.e., in symbols $\mathcal{P}(\neg\mathcal{P}\varphi \wedge \varphi)$, which is not the same thing as permitting us to do something that is forbidden, in symbols $\mathcal{P}\varphi \wedge \neg\mathcal{P}\varphi$, which is contradictory.

Permitting that something that is forbidden be the case, i.e., $\mathcal{P}(\neg\mathcal{P}\varphi \wedge \varphi)$, suggests that we are free to do φ even though it is forbidden. Note that the formula $\mathcal{P}(\neg\mathcal{P}\varphi \wedge \varphi)$ is consistent in the deontic logic D, and hence not at all like the contradictory formula $\mathcal{P}\varphi \wedge \neg\mathcal{P}\varphi$.

But $\mathcal{P}(\neg\mathcal{P}\varphi \wedge \varphi)$ is disprovable in DM, its negation being equivalent to $\mathcal{O}(\mathcal{O}\neg\varphi \rightarrow \neg\varphi)$, which states that it ought to be that what is forbidden is not the case. Can one have free will in an ethically ideal world? If not, then *ought* the real world be ethically ideal? Does free will require that it be permitted that we can do something that is forbidden in the sense of $\mathcal{P}(\neg\mathcal{P}\varphi \wedge \varphi)$?

Probably the best approach toward answering these questions might be to consider a wider framework of mixed modalities where in addition to the deontic logic DM in which $\neg\mathcal{P}(\neg\mathcal{P}\varphi \wedge \varphi)$ is a theorem we also have an alethic modal logic in which $\Diamond(\neg\mathcal{P}\varphi \wedge \varphi)$ is consistent if not also provable. Thus although $\neg\mathcal{P}(\neg\mathcal{P}\varphi \wedge \varphi)$ says that it is forbidden that we do something that is forbidden, the formula $\Diamond(\neg\mathcal{P}\varphi \wedge \varphi)$ says that we *can* do something that is forbidden, which suggests that we have free will to do so. We will not go into this mixed approach here, however.

6 The Deontic Logic *DBr*

The alethic modal logic Br has as its special axiom the thesis that what is the case is necessarily possible: $\varphi \rightarrow \Box\Diamond\varphi$. The deontic counterpart is the claim that what is the case (no matter how horrible or evil) ought to be permitted, which is obviously false, and therefore unacceptable. A revised counterpart is that in an ideal world it ought to be that what is the case ought to be permitted. The relevant alternative, in other words, is: $\mathcal{O}(\varphi \rightarrow \mathcal{O}\mathcal{P}\varphi)$. We will retain the principle thesis of DM, stipulating that an ideal world ought to be the case, which is implicitly assumed by the DBr thesis $\mathcal{O}(\varphi \rightarrow \mathcal{O}\mathcal{P}\varphi)$, which means that DBr is therefore an extension of DM.

Axiom1: $\vdash_{DBr} \varphi$ if φ is a tautology,

Axiom 2: $\vdash_{DBr} \mathcal{O}(\varphi \rightarrow \psi) \rightarrow (\mathcal{O}\varphi \rightarrow \mathcal{O}\psi)$.

Axiom 3: $\vdash_{DBr} \mathcal{O}\varphi \rightarrow \mathcal{P}\varphi$.

Axiom 4: $\vdash_{DBr} \mathcal{O}(\mathcal{O}\varphi \rightarrow \varphi)$.

Axiom 5: $\vdash_{DBr} \mathcal{O}(\varphi \rightarrow \mathcal{O}\mathcal{P}\varphi)$.

The following are theorems of DBr not provable in DM. These theorems indicate more complex nested connections between the deontic operators:

1. $\vdash_{DBr} \mathcal{O}\mathcal{P}\mathcal{O}\varphi \rightarrow \mathcal{O}\varphi$.

2. $\vdash_{DBr} \mathcal{O}(\mathcal{P}\mathcal{O}\varphi \rightarrow \mathcal{O}\mathcal{P}\varphi)$.

7 The Deontic Logic *DS4*

The principle thesis of the alethic modal logic $S4$ is the thesis that what is necessary is not contingently necessary but necessarily necessary: $\Box\varphi \rightarrow \Box\Box\varphi$. The deontic counterpart is the thesis that that one does what one ought to do only if one ought to do what one ought to do: $\mathcal{O}\varphi \rightarrow \mathcal{O}\mathcal{O}\varphi$. We will retain as part of $DS4$ the principle thesis of DM, namely $\mathcal{O}(\mathcal{O}\varphi \to \varphi)$, but not $\mathcal{O}(\varphi \to \mathcal{O}\mathcal{P}\varphi)$, the principle thesis of DBr.

Axiom1: $\vdash_{DS4} \varphi$ if φ is a tautology,

Axiom 2: $\vdash_{DS4} \mathcal{O}(\varphi \to \psi) \to (\mathcal{O}\varphi \to \mathcal{O}\psi)$.

Axiom 3: $\vdash_{DS4} \mathcal{O}\varphi \to \mathcal{P}\varphi$.

Axiom 4: $\vdash_{DS4} \mathcal{O}(\mathcal{O}\varphi \to \varphi)$.

Axiom 5: $\vdash_{DS4} \mathcal{O}\varphi \to \mathcal{O}\mathcal{O}\varphi$.

Some theorems of $DS4$ that indicate new logical nested connections between the sentential and deontic operators:

1. $\vdash_{DS4} \mathcal{O}\varphi \leftrightarrow \mathcal{O}\mathcal{O}\varphi$.

2. $\vdash_{DS4} \mathcal{P}\varphi \leftrightarrow \mathcal{P}\mathcal{P}\varphi$.

3. $\vdash_{DS4} \mathcal{O}\varphi \leftrightarrow \mathcal{O}\mathcal{P}\mathcal{O}\varphi$.

4. $\vdash_{DS4} \mathcal{O}(\varphi \to \psi) \to \mathcal{O}(\mathcal{O}\varphi \to \mathcal{O}\psi)$.

5. $\vdash_{DS4} \mathcal{O}\varphi \to \mathcal{P}\mathcal{O}\varphi$.

8 The Deontic Logic *DS4.2*

In all of the systems considered so far it is consistent to have a state of affairs φ be both permitted to be obligatory and permitted to be forbidden, i.e., $\mathcal{P}\mathcal{O}\varphi \wedge \mathcal{P}\mathcal{O}\neg\varphi$ might possibly be true in the real world as determined by each of the previous deontic logics. To exclude $\mathcal{P}\mathcal{O}\varphi \wedge \mathcal{P}\mathcal{O}\neg\varphi$, which says that φ is permitted to be both obligatory and forbidden, it suffices to add to $DS4$ the deontic thesis $\mathcal{P}\mathcal{O}\varphi \to \mathcal{O}\mathcal{P}\varphi$, which is equivalent to the negation of $\mathcal{P}\mathcal{O}\varphi \wedge \mathcal{P}\mathcal{O}\neg\varphi$. The new thesis $\mathcal{P}\mathcal{O}\varphi \to \mathcal{O}\mathcal{P}\varphi$ says that φ is permitted to be obligatory only if it is obligatory that it be permitted, and hence only if it is not permitted to be forbidden.

Adding $\mathcal{P}\mathcal{O}\varphi \to \mathcal{O}\mathcal{P}\varphi$ to the $DS4$ results in the counterpart of the alethic modal logic $S4.2$. The modal thesis of $S4.2$, namely $\Diamond\Box\varphi \to \Box\Diamond\varphi$, allows a commutation of $\Diamond\Box$ with $\Box\Diamond$ in one direction. Adding the deontic counterpart of this modal thesis to $DS4$ gives us the deontic logic $DS4.2$.

Axiom1: $\vdash_{DS4.2} \varphi$ if φ is a tautology,

Axiom 2: $\vdash_{DS4.2} \mathcal{O}(\varphi \to \psi) \to (\mathcal{O}\varphi \to \mathcal{O}\psi)$.

Axiom 3: $\vdash_{DS4.2} \mathcal{O}\varphi \to \mathcal{P}\varphi$.

Axiom 4: $\vdash_{DS4.2} \mathcal{O}(\mathcal{O}\varphi \to \varphi)$.

Axiom 5: $\vdash_{DS4.2} \mathcal{O}\varphi \to \mathcal{O}\mathcal{O}\varphi$.

Axiom 6: $\vdash_{DS4.2} \mathcal{P}\mathcal{O}\varphi \to \mathcal{O}\mathcal{P}\varphi$.

The following are theorems of $DS4.2$ indicating nested laws regarding the deontic operators not provable in previous systems. Theorem 2, in particular, says that φ is permitted to be obligatory only if it is permitted *simpliciter*:

1. $\vdash_{DS4.2} \mathcal{PO}\varphi \to \mathcal{OPO}\varphi$.

2. $\vdash_{DS4.2} \mathcal{PO}\varphi \to \mathcal{P}\varphi$.

9 The Deontic Logic *DS4.3*

In addition to allowing states of affairs that are both permitted to be obligatory and permitted to be forbidden, the deontic logics DBr, and $DS4$ also allow that there are states of affairs φ and ψ that are both permissible and yet it is forbidden that one be the case while the other is permitted. That is, for some formulas φ, ψ,

$$\mathcal{P}\varphi \wedge \mathcal{P}\psi \wedge \neg\mathcal{P}(\varphi \wedge \mathcal{P}\psi) \wedge \neg\mathcal{P}(\psi \wedge \mathcal{P}\varphi),$$

is consistent. This formula, by theorem 6 for DKr, is equivalent to

$$\mathcal{P}\varphi \wedge \mathcal{P}\psi \wedge \neg\mathcal{P}[(\varphi \wedge \mathcal{P}\psi) \vee (\psi \wedge \mathcal{P}\varphi)],$$

which is consistent even in $DS4.2$. It is not clear what meta-ethical issues would warrant allowing for this possibility. In any case, if we wish to reject it what is needed is the deontic correlate of the principle thesis of the modal logic $S4.3$, namely:

$$\mathcal{P}\varphi \wedge \mathcal{P}\psi \to \mathcal{P}[(\varphi \wedge \mathcal{P}\psi) \vee (\psi \wedge \mathcal{P}\varphi)].$$

Adding this thesis to $DS4$ gives us $DS4.3$.

Axiom1: $\vdash_{DS4.3} \varphi$ if φ is a tautology,

Axiom 2: $\vdash_{DS4.3} \mathcal{O}(\varphi \to \psi) \to (\mathcal{O}\varphi \to \mathcal{O}\psi)$.

Axiom 3: $\vdash_{DS4.3} \mathcal{O}\varphi \to \mathcal{P}\varphi$.

Axiom 4: $\vdash_{DS4.3} \mathcal{O}(\mathcal{O}\varphi \to \varphi)$.

Axiom 5: $\vdash_{DS4.3} \mathcal{O}\varphi \to \mathcal{OO}\varphi$.

Axiom 6: $\vdash_{DS4.3} \mathcal{P}\varphi \wedge \mathcal{P}\psi \to \mathcal{P}[(\varphi \wedge \mathcal{P}\psi) \vee (\psi \wedge \mathcal{P}\varphi)]$.

The following thesis of $DS4.2$ is a theorem of $DS4.3$, which shows that $DS4.3$ is an extension of $DS4.2$:

$\vdash_{DS4.3} \mathcal{PO}\varphi \to \mathcal{OP}\varphi$.

Note: Although $DS4.2$ is a proper subsystem of $DS4.3$, nevertheless it turns out that the result of adding the principle thesis of $DS4.2$ to $DBr + DS4$ is equivalent to $DBr + DS4.3$.

Theorem: $DBr + DS4.2$ is equivalent to $DBr + DS4.3$.

10 The Deontic Logic *DS5*

The final thesis of alethic modal logic to consider is the principle that what is possible is necessarily possible. The deontic counterpart of this is the thesis that what is permissible ought to be permissible: $\mathcal{P}\varphi \to \mathcal{OP}\varphi$. In alethic modal logic the $S4$ thesis $\Box\varphi \to \Box\Box\varphi$ is provable in $S5$; but the proof depends on the modal thesis $\Box\varphi \to \varphi$ of M, the deontic counterpart of which is not acceptable, as we have already noted. As a result, the $DS4$ thesis $\mathcal{O}\varphi \to \mathcal{O}\mathcal{O}\varphi$ is not provable in $DS5$, which is why we retain it here as an axiom of $DS5$:

Axiom1: $\vdash_{DS5} \varphi$ if φ is a tautology,

Axiom 2: $\vdash_{DS5} \mathcal{O}(\varphi \to \psi) \to (\mathcal{O}\varphi \to \mathcal{O}\psi)$.

Axiom 3: $\vdash_{DS5} \mathcal{O}\varphi \to \mathcal{P}\varphi$.

Axiom 4: $\vdash_{DS5} \mathcal{P}\varphi \to \mathcal{O}\mathcal{P}\varphi$.

Axiom 5: $\vdash_{DS5} \mathcal{O}\varphi \to \mathcal{O}\mathcal{O}\varphi$.

Note that by theorem 15 for DKr and the $DS5$ axiom 4, the formula $\mathcal{O}(\varphi \to \mathcal{P}\varphi)$, which is equivalent to the DM thesis $\mathcal{O}(\mathcal{O}\varphi \to \varphi)$, is provable in $DS5$. In other words, DM is a subsystem of $DS5$. Similarly, by theorem 15 for DKr and the $DS5$ axiom 5, the DBr thesis $\mathcal{O}(\varphi \to \mathcal{O}\mathcal{P}\varphi)$ is provable in $DS5$. Hence DBr is a subsystem of $DS5$ as well. Finally, the principle thesis of $DS4.3$ (and therefore of $DS4.2$ as well) is also provable in $DS5$. The deontic logic $DS5$, in other words, contains all of the previous systems.

1. $\vdash_{DS5} \mathcal{O}(\mathcal{O}\varphi \to \varphi)$.
2. $\vdash_{DS5} \mathcal{O}(\varphi \to \mathcal{O}\mathcal{P}\varphi)$.
3. $\vdash_{DS5} \mathcal{O}(\mathcal{O}\varphi \to \mathcal{O}\mathcal{P}\varphi)$.
4. $\vdash_{DS5} \mathcal{P}\varphi \wedge \mathcal{P}\psi \to \mathcal{P}[(\varphi \wedge \mathcal{P}\psi) \vee (\psi \wedge \mathcal{P}\varphi)]$.

Note: The alethic modal logic $S5$ is well-known to be equivalent to the combined alethic modal logic $Br + S4$. The related deontic logics are not equivalent, however. The $DS5$ axiom $\mathcal{P}\varphi \to \mathcal{O}\mathcal{P}\varphi$ in particular is not provable in $DBr+DS4$. What we have instead is the equivalence of $DS5$ with the combined deontic logics DBr and $DS4.2$. Of course, because $DS4.3$ contains $DS4.2$, it follows that $DS5$ is also equivalent to $DBr + DS4.3$.

Theorem $DS5$ is equivalent to both $DBr + DS4.2$ and $DBr + DS4.3$.

11 Deontic World Systems

In constructing a possible-worlds semantics for deontic logic we will follow the way it is commonly done in alethic modal logic. That means we begin with an ordered collection of possible worlds, which we call a world system. One world w_2 is related to another world w_1 in such a system if it is a permissible alternative to that world, i.e., if what is permissible in w_1 is true in w_2. The semantic idea is that a formula of the form $\mathcal{P}\varphi$ will be true in w_1 if, and only if, there is a world w_2 in which φ is true that is a permissible alternative to w_1. Similarly, $\mathcal{O}\varphi$ will be true in a world of the system if, and only if, φ is true in every world of the system that is a permissible alternative to the given world.

We can represent each possible world on this level of analysis by a truth-value assignment that assigns to each propositional variable a truth value, truth or falsehood. The truth values of conjunctions, disjunctions, conditionals, negations, etc., are then defined in the usual truth-functional manner. Finally the truth value of a formula of the form $\mathcal{O}\varphi$ with respect to such an assignment t is truth if φ is true in every permissible world (truth-value assignment) alternative to t, and false otherwise; and a formula of the form $\mathcal{P}\varphi$ is true with respect to t if there is a permissible world (truth-value assignment) alternative to t in which φ is true.

Definition: $\mathfrak{A} = \langle R, t_{k\in W}\rangle$ is a *world system* iff

1. W is a set of possible worlds,
2. R is a relation on W, i.e., $R \subseteq W \times W$,and
3. for each k in W, t_k is a truth-value assignment to the propositional variables.

Truth in a world system at a given world of that system can be defined in the obvious way. *Validity* in a world system is then definable as truth at every world in that world system; and validity *simpliciter* is defined as validity in every world system.

As already noted, the minimal alethic modal logic Kr is complete with respect to validity *simpliciter*, and, since there is no difference between KR and DKr except for having $\mathcal{O}$ occur wherever $\Box$ occurs in a formula, the same applies to the deontic logic DKr. The minimality of DKr, accordingly, does not structurally distinguish the deontological modalities $\mathcal{O}$ and $\mathcal{P}$ from the alethic modalities of $\Box$ and $\Diamond$. We will redefine the notion of a world system so that completeness theorems can be shown for the more interesting extensions of DKr. We will refer to the redefined notion as that of a deontic world system.

In particular, we will assume that there is an initial or first possible world in a deontic world system, which we may consider to be the real world. Each of the other worlds in such a system are then either permissible alternatives to the real world or are permissible-alternative descendants of the real world, i.e., they are permissible worlds of a permissible world, or they are permissible worlds of permissible worlds that are permissible worlds of the real world, etc.

Definition: $\mathfrak{A} = \langle i, R, t_{k\in W}\rangle$ is a *deontic world system* iff

1. W is a set of possible worlds,
2. $i \in W$,
3. R is a relation on W, i.e., $R \subseteq W \times W$,
4. for k in W, t_k is a truth-value assignment to the propositional variables,
5. every world k in W has an R-alternative world j in W, and
6. for each world k in W there is a natural number n such that k is n permissible R-steps away from i.

Where $\mathfrak{A} = \langle i, R, t_{k\in W}\rangle$ is a deontic world system, we take i to represent the actual world. Clause 5 is required because $\mathcal{O}\varphi \to \mathcal{P}\varphi$ is an axiom in the deontic logic D and therefore provable in all of the deontic systems containing D. Given any world j in W, $\mathcal{O}(\varphi \vee \neg\varphi)$ will be true at j, and therefore so will $\mathcal{P}(\varphi \vee \neg\varphi)$, which means that there must a world k that is a permissible alternative to j.

Truth in a deontic world system at a given world is defined as follows.

Definition: If $\mathfrak{A} = \langle i, R, t_{i \in W} \rangle$, $\mathfrak{A}$ is a deontic world system and for each k in W,
1. If φ is atomic formula $\mathbf{p}_n$, then φ is true in $\mathfrak{A}$ at k iff t_k assigns truth to $\mathbf{p}_n$,
2. $\neg\varphi$ is true in $\mathfrak{A}$ at k iff φ is not true, i.e., false in $\mathfrak{A}$ at k,
3. $(\varphi \to \psi)$ is true in $\mathfrak{A}$ at k iff either φ is false or ψ is true in $\mathfrak{A}$ at k, and
4. $\mathcal{O}\varphi$ is true in $\mathfrak{A}$ at k iff for all $j \in W$, if j is a permissible R-alternative to k, i.e., $\langle k, j \rangle \in R$, then φ is true in $\mathfrak{A}$ at j.

Deontic validity in a deontic world system is defined as truth at every world in that system, and deontic entailment is defined similarly.

Definition: If $\mathfrak{A} = \langle i, R, t_i \rangle_{i \in W}$, $\mathfrak{A}$ is a deontic world system, and k is a world in W, and $\Gamma \cup \{\varphi\}$ is a set of formulas, then
(1) φ is *deontically valid in* $\mathfrak{A}$, in symbols $\models_{\mathfrak{A}} \varphi$, iff φ is true at every world in $\mathfrak{A}$; and
(2) Γ *deontically entails* φ in $\mathfrak{A}$, in symbols $\Gamma \models_{\mathfrak{A}} \varphi$, iff for each world k of $\mathfrak{A}$, if ψ is true at k for every ψ in Γ, then φ is true in $\mathfrak{A}$ at k as well.

A deontic logic Σ, we will say, strongly characterizes a class A of deontic world systems if derivability in Σ coincides with deontic entailment in every world in A; and Σ characterizes A if provability in Σ coincides with validity in every world in A.

Definition: If Σ is a (propositional) deontic logic and A is a class of world systems, then
(1) Σ strongly *characterizes* A if,and only if, for all sets of formulas Γ and all formulas φ, $\Gamma \vdash_\Sigma \varphi$ iff for all deontic world systems $\mathfrak{A}$ in A, $\Gamma \models_{\mathfrak{A}} \varphi$;
(2) Σ *characterizes* A if,and only if, for all formulas φ, $\vdash_\Sigma \varphi$ iff for all deontic world systems $\mathfrak{A}$ in A, $\models_{\mathfrak{A}} \varphi$.

Lemma: If a deontic logic strongly characterizes a class of deontic world systems, then it also characterizes that class.

12 Completeness Theorems

We have noted that every world in a deontic world system has a permissible alternative, and, other than the real world, every world has a world that is permissible alternative to it, even if that alternative is only itself. No possible world in such a system can be isolated, accordingly, except in the sense of being its own permissible alternative. But note that a world that is a permissible alternative of itself is an ideal world, because then whatever is permissible in that world is the case in that world, and whatever ought to be the case in that world is also the case in that world.

Our first completeness theorem concerns the deontic system D, which is minimal in that it only requires that every world have a permissible alternative.[1]

Theorem 1: (1) For all formulas φ and sets of formulas Γ, $\Gamma \vdash_D \varphi$ iff for all deontic world systems $\mathfrak{A}$, $\Gamma \models_{\mathfrak{A}} \varphi$; and (therefore) (2) the deontic system D strongly characterizes the class of deontic world systems.

The system D is minimal as a deontic logic. But, as we noted in section 5, D allows for the permissibility of a forbidden state of affairs. In other words, $\mathcal{P}(\neg\mathcal{P}\varphi \wedge \varphi)$ is consistent in D, because $\mathcal{O}(\mathcal{O}\neg\varphi \to \neg\varphi)$ is not a theorem of D, which means that its negation, which is equivalent to $\mathcal{P}(\neg\mathcal{P}\varphi \wedge \varphi)$ can be true for some formula φ.

What is needed to correct this is the thesis $\mathcal{O}(\mathcal{O}\varphi \to \varphi)$, which semantically stipulates that only ethically ideal worlds should be permissible. A permissible world, in other words, should be one in which only what ought to be the case in that world is in fact the case.

Definition: If $\mathfrak{A}$ is a deontic world system and k is a world in $\mathfrak{A}$, then:

(1) k is an *ethically ideal world* if, and only if, for all formulas φ, $\mathcal{O}(\mathcal{O}\varphi \to \varphi)$ is true at k in $\mathfrak{A}$; and

(2) $\mathfrak{A}$ is *ethically ideal world system* if, and only if, every permissible alternative in $\mathfrak{A}$ is ethically ideal, i.e., iff for each world k in $\mathfrak{A}$, if k is a permissible alternative of some world j in $\mathfrak{A}$, then k is ethically ideal.

Lemma: If $\mathfrak{A} = \langle i, R, t_{k\in W}\rangle$ is a *deontic world system, then* $\mathfrak{A}$ is ethically ideal if, and only if, the relation R of permissible alternatives in $\mathfrak{A}$ is reflexive, i.e., iff for each world k in $\mathfrak{A}$, if k is a permissible alternative of some world j in $\mathfrak{A}$, then k is a permissible alternative of itself, i.e., $\langle k, k\rangle \in R$.

Theorem 2: The deontic system DM strongly characterizes the class of ethically ideal deontic world system, i.e., the class of deontic world systems that are reflexive in their range.

It might be argued that the above characterization of ethical ideality is not quite correct. The logic DM does characterize ethical ideality in the sense defined, but, because $\mathcal{O}(\varphi \to \mathcal{O}\mathcal{P}\varphi)$ is not provable in DM, its negation $\mathcal{P}(\varphi \wedge \neg\mathcal{P}\mathcal{O}\varphi)$ is consistent in DM. Therefore, by theorem 2 and the definition of ethical ideality, $(\varphi \wedge \neg\mathcal{P}\mathcal{O}\varphi)$ is true in some ethically ideal world. What this formula says is that some state of affairs that is forbidden to be obligatory is the case in an ethically ideal world, and that seems counter-intuitive.

Of course, it is unreasonable to require that whatever is the case (no matter how bad it is) in the real world ought therefore to be permitted to be the case. But in an ethically ideal world one might maintain that only what ought to be permitted is in fact the case. That means that the relation of alternative

[1] Proofs of completeness theorems in this section are entirely similar to those in chapter 6 of *Modal Logic: An Introduction to its Syntax and Semantics*, Oxford University Press, 2008, by Nino B. Cocchiarella and Max A. Freund.

permissibility of any one ethically ideal world with respect to another must be symmetrical, i.e., that each must be permissible with respect to the other. Symmetric permissibility between ethically ideal worlds that are permissible alternatives, i.e., that are in the range of the permissibility relation, is logically enforced by the deontic thesis $\mathcal{O}(\varphi \to \mathcal{O}\mathcal{P}\varphi)$ of the logic DBr.

Theorem 3: The deontic logic DBr strongly characterizes the class of deontic world systems that are both reflexive and symmetric in the range of their alternative permissibility relations.

Another problem with ethically ideality as defined above and characterized by the deontic logic DM is that it allows a forbidden state of affairs to be permissibly permissible. That is, $(\neg\mathcal{P}\varphi \wedge \mathcal{P}\mathcal{P}\varphi)$ is consistent in DM for some formula φ. In other words, the negation of this formula, namely $\mathcal{O}\varphi \to \mathcal{O}\mathcal{O}\varphi$, which is equivalent to $\mathcal{P}\mathcal{P}\varphi \to \mathcal{P}\varphi$, is not provable in DM.

Intuitively, no permissible state of affairs should be both denied and yet permissibly permissible. In other words, any world that is a permissible alternative of a permissible alternative of a world k should itself be a permissible alternative of k. That means that the permissible alternative relation should be transitive. Transitivity is what is required by the deontic thesis $\mathcal{O}\varphi \to \mathcal{O}\mathcal{O}\varphi$ of $DS4$.

Theorem 4: The deontic logic $DS4$ strongly characterizes the class of deontic world systems that are transitive and reflexive in the range, i.e., $DS4$ strongly characterizes the class of transitive ethically ideal deontic world systems.

A transitive ethically ideal deontic world system is an appropriate type of semantic structure for deontic logic in that each permissible world of such a structure is not only ethically ideal but also a permissible alternative of the real world of that structure. Thus, $DS4$ allows us to view a deontic world system as one in which all of the permissible alternatives of the real world are ethically idealized alternatives of the real world. What is permissible in the real world is so because it is actually the case in at least one of the idealized alternatives of the real world. Also, what is forbidden in the real world of such a system is what is not realized in any ethically idealized alternative of the real world. In addition, those and only those states of affairs that are realized in every ethically idealized alternative of the real world are obligatory in the real world.

Combining the desirable features of the transitive deontic world systems that are characterized by $DS4$ with the symmetrical systems characterized by DBr results in those deontic systems in which the permissible-alternative relation is an equivalence relation on the set of permissible alternatives, i.e., on the range of permissible-alternative relation. Such an equivalence relation partitions the worlds that are permissible alternatives into equivalence cells in which all of the worlds in such a cell are permissible alternatives to all of the other worlds in that cell.

Theorem 5: The combined system $DBr + DS4$ strongly characterizes the class of deontic world systems that are partitioned in the range of their permissible-alternative relations.

Each of the deontic logics DBr, $DS4$, and their union $DBr + DS4$ allow for the possibility that some state of affairs φ is both permitted to be obligatory and permitted to be forbidden, i.e., both $\mathcal{PO}\varphi$ and $\mathcal{PO}\neg\varphi$ can be true in the real world of a deontic world system $\mathfrak{A}$ that is partitioned in its range. That is, we could have $\mathcal{PO}\varphi \wedge \mathcal{PO}\neg\varphi$ be true in the real world of $\mathfrak{A}$, so that $\mathcal{O}\varphi$ will then be true in all of the worlds of one of the cells of the partition while $\mathcal{O}\neg\varphi$ is true in all of the worlds of another cell of the partition.

To exclude the possibility of a state of affairs φ being both permitted to be obligatory and permitted to be forbidden, i.e., to exclude $\mathcal{PO}\varphi \wedge \mathcal{PO}\neg\varphi$ as possibly being true in the real world, requires that we take the alternative-permissibility relation to be connectable in its range, i.e., to be such that any two permissible alternative worlds have a world that is a permissible alternative to both. The deontic principle $\mathcal{PO}\varphi \rightarrow \mathcal{OP}\varphi$, the negation of which is equivalent to $\mathcal{PO}\varphi \wedge \mathcal{PO}\neg\varphi$, excludes this possibility. This principle is the special axiom of the deontic logic $DS4.2$. Adding this principle to $DBr + DS4$ results in $DBr + DS4.2$, which we noted in section 9 is equivalent to $DS5$.

Definition: If $\mathfrak{A} = \langle i, R, t_i \rangle_{i \in W}$ is a deontic world system, then $\mathfrak{A}$ *is connectable in its range* if, and only if, for all worlds i, j of $\mathfrak{A}$, there is a world k of $\mathfrak{A}$ that is a permissible alternative of both i and j, i.e., there is a $k \in W$ such that $\langle i, k \rangle \in R$ and $\langle j, k \rangle \in R$.

Theorem 6: $DS4.2$ strongly characterizes the class of deontic world systems that are reflexive and connectable their range; that is, $DS4.2$ strongly characterizes the class of ethically ideal deontic world systems in which any two permissible worlds have a common permissible alternative.

As noted in section 9, the formula,

$$\mathcal{P}\varphi \wedge \mathcal{P}\psi \wedge \neg\mathcal{P}[(\varphi \wedge \mathcal{P}\psi) \vee (\psi \wedge \mathcal{P}\varphi)],$$

which, by theorem 6 of DKr, is equivalent to

$$\mathcal{P}\varphi \wedge \mathcal{P}\psi \wedge \neg\mathcal{P}(\varphi \wedge \mathcal{P}\psi) \wedge \neg\mathcal{P}(\psi \wedge \mathcal{P}\varphi),$$

is consistent not only in the deontic logics DBr and $DS4$ but also in $DS4.2$. The consistency of this formula allows for there being states of affairs that are jointly permissible and yet it is forbidden that one actually be the case while the other is permitted. In a deontic world system that is partitioned in its range and that has two permissibility cells, for example, we could have φ true in a world in one of the cells while ψ is forbidden in that world, and therefore false in all of the worlds of that cell, and yet ψ is true and φ is forbidden in one of the worlds of the other cell and hence φ is false in all of the worlds of that cell.

In other words, both $\mathcal{P}\varphi$ and $\mathcal{P}\psi$ will be true in the real world of such a deontic world system, and yet $\neg\mathcal{P}(\varphi \wedge \mathcal{P}\psi)$ and $\neg\mathcal{P}(\psi \wedge \mathcal{P}\varphi)$ can also true in that world.

To exclude such pairs of permissible states of affairs requires that permissible alternatives in a deontic world system be permissibly comparable, i.e., at least one of two is a permissible alternative of the other. What is required, accordingly, is that the permissible alternative relation be *quasi-connected in its range*, i.e., if i and j are permissible alternatives of the same world k, then either $i = j$ or i is a permissible alternative of j or j is a permissible alternative of i.

Theorem 7: The deontic logic $DS4.3$ strongly characterizes the class of transitive deontic world systems that are both reflexive and quasi-connected in their range, i.e. $DS4.4$ strongly characterizes the class of transitive ethically ideal deontic world systems in which any two permissible worlds are permissibly comparable.

In an earlier remark we noted that the special deontic principle of $DS4.2$ affected the class of deontic world systems partitioned in their range so as to reduce them to but one permissibility cell, i.e., to those in which permissibility between permissible alternatives is a universal relation. Clearly, because $DS4.3$ contains $DS4.2$, a similar observation hold for the principle deontic thesis for $DS4.3$.

Moreover, because $DS5$ is equivalent to both $DBr + DS4.2$ and $DBr + DS4.3$, this special subclass of the deontic world systems partitioned in their range is also strongly characterized by $DS5$.

Definition: $\mathfrak{A} = \langle i, R, t_i \rangle_{i \in W}$ is a deontic world system that is *universally related in its range* if, and only if, $\mathfrak{A}$ is a deontic world system and for all worlds i, j in the range of R, i is a permissible alternative of j, i.e., $\langle i, j \rangle \in R$.

Theorem 8: $DS5$ strongly characterizes the class of deontic world systems that are universally related in their range.

Deontic world systems that are universally related in their range are of special interest in deontic logic because they have all of the desirable features noted above and none of the undesirable ones that we noted for the weaker systems. In particular, a deontic world system that is universally related in its range consists of a real world that need not itself be ethically ideal but that nevertheless has a collection of ethically idealized worlds as permissible alternatives that are permissible alternatives of each other.

It can also be shown that the ethically ideal worlds in such a deontic world system are "indiscernible" in their deontically closed formulas, i.e., formulas in which every occurrence of a propositional letter lies within the scope a deontic operator; and therefore they are worlds that are indiscernible *simpliciter* if they have the same extensional, non-deontic facts. What is permissible in the real world of such a world system then amounts to what is true in an ethically ideal world, and what is obligatory in the real world amounts to what is true in each of the real world's ethically idealized alternatives.