Logical Necessity Based on Carnap's Criterion of Adequacy

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Abstract

A semantics for logical necessity, based on Carnap's criterion of adequacy, is given with respect to the ontology of logical atomism. A calculus for sentential (propositional) modal logic is described and shown to be complete with respect to this semantics. The semantics is then modified in terms of a restricted notion of 'all possible worlds' in the interpretation of necessity and shown to yield a completeness theorem for the modal logic $S5$. Such a restricted notion introduces material content into the meaning of necessity so that, in addition to atomic facts, there are "modal facts" that distinguish one world from another.

In a well-known paper on "The Philosophical Significance of Modal Logic," Gustav Bergmann suggested that one might make sense of propositional modal logic in terms of a four-valued matrix in which the values are taken as *necessary truth*, *contingent truth*, *contingent falsehood*, and *necessary falsehood*, respectively.[1] In this way, Bergmann claimed, "one *might* ... conceivably arrive at an adequate explication, very much in the style of truth tables, of what could be meant by calling logical truths necessary" (p.483). Such an explication could not succeed, however, because it had already been shown that no finite matrix is characteristic of modal logic—or at least not of any of the so-called normal systems of Lewis and Langford.[2] What this showed, according to Bergmann, was that modal logic has no philosophical significance.

This conclusion is not only wrong, but wrongly based as well. This is because the result in question was proven in terms of a certain type of matrix known as a Henle matrix, which means that it applies to systems that do not validate, for any positive integer n, the statement that there are at most n propositions—where by a proposition we mean the kind of entity that can be associated with a set of possible worlds (or the characteristic function of such a set). That is, the result applies to systems for which it is not assumed (nor rejected for that matter) that there are only a finite number of possible worlds. This, we maintain, is as it should be—or, at least, it certainly is as it should be in the case of logical necessity as the modal counterpart of the semantic notion of logical truth, which is the only notion of necessity considered by Bergmann. In this regard, the result is not about the number of truth values that a proposition

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[1] G. Bergmann 1960.
[2] See J. Dugundji 1940.

might have—which in a Henle matrix is still just two, namely, truth and falsity—but about the number of possible worlds in which a proposition might be true or false.

What can be meant by calling logical truths necessary? The answer consists in constructing an appropriate semantics for sentential (propositional) modal logic, where $\Box$ represents logical necessity and $\Diamond$ represents logical possibility, and then describe a modal logic that can capture this semantics by proving a completeness theorem for the logic.[3] That Bergmann's finite-matrix proposal cannot succeed does not show that no semantics can. Indeed. in what follows we will construct a semantics for logical necessity based on Rudolf Carnap's criterion of adequacy and the metaphysical framework of logical atomism, a semantics, we maintain, that provides a clear and precise account of the connection between logical truth and logical necessity—at least with respect to this kind of metaphysical framework.[4] We will assume throughout that possibility, as represented by $\Diamond$, is definable (analyzable) in terms of negation and necessity, i.e., that $\Diamond$ can be taken as an abbreviation of $\neg\Box\neg$.

1 The Syntax of Propositional Modal Logic

As primitive symbols of propositional modal logic we will use '$\neg$' as the symbol for negation, '$\rightarrow$' as the symbol for the (material) conditional, and, as already noted, '$\Box$' as the symbol for (logical) necessity. The logical constants '$\wedge$', '$\vee$', '$\leftrightarrow$', and '$\Diamond$' for conjunction, disjunction, biconditionality, and possibility are defined in the metalanguage, which we take to be ZF set theory, as follows (with φ, ψ, and χ as metalanguage variables for formulas):

1. $(\varphi \wedge \psi) =_{df} \neg(\varphi \rightarrow \neg\psi)$

2. $(\varphi \vee \psi) =_{df} (\neg\varphi \rightarrow \psi)$

3. $(\varphi \leftrightarrow \psi) =_{df} [(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)]$

4. $\Diamond\varphi =_{df} \neg\Box\neg\varphi$.

We assume the usual conventions about sometimes deleting or dropping parentheses. In particular, we assume that '$\wedge$' and '$\vee$' apply before '$\rightarrow$' and '$\leftrightarrow$'. We take the (potentially) infinite sequence $P_0, P_1, ..., P_n, ...,$ (for each $n \in \omega$) as sentence letters (propositional variables).[5] Because the conditional, negation and necessity signs are the only primitive logical constants, we call the formulas

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[3]Or show, as is the case in quantified modal logic for full predicate logic, that the semantics cannot be completely captured because it yields an essential incompleteness theorem. See Cocchiarella 1975 for such a result. Such an incompleteness theorem does not nullify the semantics, though this takes us into considerations that do not concern us here.

[4]There are reasons to think that no other sort of metaphysical framework can succeed in adequately explaining the connection between logical truth and logical necessity. This is not to say, however, that other frameworks cannot account for notions of necessity other than logical necessity.

[5]We understand ω to be the set of natural numbers.

of the resulting sentential (propositional) modal logic *modal CN-formulas*, which we define inductively as follows:

Definition 1 *φ is a modal CN-formula, in symbols, $\varphi \in FM$, if, and only if, φ belongs to every set K to which P_n belongs, for $n \in \omega$, and which is closed under the formation of conditionals, negations and necessity; i.e., $FM =_{df} \cap K[P_n \in K$, for all $n \in \omega$, and for all $\varphi, \psi \in K$, $\neg\varphi$, $(\varphi \rightarrow \psi)$, and $\Box\varphi \in K]$.*

The following induction principle is an immediate set-theoretic consequence of this definition.

Theorem 2 *(Induction principle for FM):*
If (1) for each $n \in \omega$, $P_n \in K$,
(2) for all $\varphi \in K$, $\neg\varphi \in K$,
(3) for all $\varphi, \psi \in K$, $(\varphi \rightarrow \psi) \in K$, and
(4) for all $\varphi \in K$, $\Box\varphi \in K$,
then $FM \subseteq K$.

2   Semantics for Logical Necessity

As already noted, a natural, intuitive formal semantics can be given for logical necessity if we base it on the metaphysical framework of logical atomism.[6] This is because a logically possible world is completely determined in logical atomism by the atomic states of affairs that obtain in that world, which is perhaps the clearest notion of a logically possible world that we can have. Thus, if each atomic sentence letter is taken to represent an atomic state of affairs, then a possible world can be represented by a distribution of truth values to all of the sentence letters, i.e., by a complete representation of the atomic states of affairs that obtain in that world as opposed to those that do not.[7] For convenience, we will represent truth by 1 and falsehood by 0. We call an assignment of 0 and 1 to the sentence letters a *truth-value assignment*.

Definition 3 *t is a truth-value assignment, in symbols $t \in V$, if, and only if, $t \in \{0,1\}^{\{P_n : n \in \omega\}}$, i.e. iff t is a function from the set of sentential letters into $\{0,1\}$.*

By a *modal-free* formula we mean a formula in which '$\Box$' does not occur, i.e., in which the only logical constants that occur are the conditional and the negations signs. For this reason we will call these formulas *CN-formulas*, the set of which is represented by 'FM_{CN}'.

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[6]For a fuller discussion of the semantics of logical necessity in the metaphysical background of logical atomism, see chapter six, "Logical Atomism and Modal Logic" and chapter seven, "Logical Atomism, Nominalism, and Modal Logic," of N. B. Cocchiarella 1987.

[7]If there are only a finite number of atomic states of affairs, then different sentence letters will be assigned the same truth value.

Definition 4 φ *is a modal-free formula, in symbols* $\varphi \in FM_{CN}$, *if, and only if,* $\varphi \in FM$ *and* $\Box$' *does not occur in* φ.

The truth-value in a possible world of a modal-free formula can of course be inductively defined in the usual way as follows. In particular, where $\varphi \in FM_{CN}$, we will read '$\models_t \varphi$' as 'φ is true in (or with respect to) t' and '$\nvDash_t \varphi$' as 'φ is not true in t'. The definition is as follows.

Definition 5 *If* $t \in V$, *then:*
(1) $\models_t P_n$ *iff* $t(P_n) = 1$,
(2) $\models_t \neg\varphi$ *iff* $\nvDash_t \varphi$, *and*
(3) $\models_t (\varphi \to \psi)$ *iff either* $\nvDash_t \varphi$ *or* $\models_t \psi$.

Now a modal-free formula is a *tautology*, i.e., a logical truth on the level of propositional logic, if, and only if, that formula is true in every truth-value assignment, i.e., in every logically possible world as understood in logical atomism. For convenience we will speak of a logically true formula as L-true.

Definition 6 *If* $\varphi \in FM_{CN}$, *then* φ *is* L-true *if, and only if, for all* $t \in V$, $\models_t \varphi$.

The notion of being a tautology, or being tautologous, can be extended to modal formulas as well so long as they are obtained from tautologous modal-free formulas by substitution of formulas for sentence letters. We define this notion as follows.

Definition 7 *If* $\varphi \in FM$, *then* φ *is tautologous if, and only if, there is a modal-free formula* $\psi \in FM_{CN}$ *such that (1)* ψ *is* L-true *(i.e., tautologous) and* φ *is obtained from* ψ *by uniformly substituting formulas for sentence letters occurring in* ψ.

Of course, taulologous modal formulas are L-true and therefore logically necessary.

Theorem 8 *If* $\varphi \in FM$ *and* φ *is tautologous, then* φ *is* L-true.

Proof. If φ is tautologous, then it is obtained from a tautologous modal-free formula ψ by uniformly substituting formulas for sentence letters. If φ were not L-true, then there would be a $t \in V$ such that $\nvDash_t \varphi$; but then, by assigning 1 or 0 to the sentence letters occurring in ψ depending on whether the formulas substituted for those sentence letters to obtain φ are true or false in t, respectively, means that $\nvDash_t \psi$, which is impossible because ψ is a modal-free tautologous formula, which means that $\models_t \psi$, for all $t \in V$. ∎

Being tautologous is not all there is to L-truth, and therefore to logical necessity, however. In particular, there are modal formulas, such as $(\Box\varphi \to \varphi)$ and $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ that are logically necessary but not tautologous,

and so we need an extended definition of truth under which these formulas are *L*-true. So the question now is how are we to extend the above definition of truth and *L*-truth so as to apply in an intuitively acceptable way to modal as well as modal-free formulas. And given such an intuitively acceptable notion of *L*-truth, another question is what propositional modal calculus, if any, captures as provable all and only the modal formulas that are *L*-true?

3   Carnap's Criterion of Adequacy

Rudolf Carnap in his book, *Meaning and Necessity,* proposed the following informal convention as a criterion of adequacy for any truth clause for logical necessity:

for any sentence φ, $\Box\varphi$ is true iff φ is *L-true*.

As restricted to modal-free formulas, i.e., for $\varphi \in FM_{CN}$, this criterion amounts exactly to φ being tautologous—i.e., by the above results, to φ being logically true—which is the kind of necessity that Bergmann intended in his criticism of modal logic. Of course, the problem Bergmann had in mind is not with the truth-conditions of $\Box\varphi$ when φ is modal free, but with $\Box\varphi$ when φ already contains occurrences of the necessity sign; but in that case the notion of *L*-truth, as it occurs in the above criterion of adequacy, presupposes that we already know what it means for a modal formula to be true in a possible world.

Note that, relative to the framework of logical atomism, where (on the propositional level of logical analysis) logically possible worlds are represented by truth-value assignments, what Carnap's criterion of adequacy amounts to for modal-free formulas, when truth is relativized to truth in a possible world, is the following:

for all $\varphi \in FM_{CN}$ and all $t \in V$, $\Box\varphi$ is true in t iff for all $t' \in V$, φ is true in t'.

In this form, Carnap's criterion is an explicit truth condition for $\Box\varphi$ when φ is modal free. If we now generalize and apply this same truth condition to formulas in general, we obtain an intuitively natural and acceptable truth condition for $\Box\varphi$ even when φ is not modal free (at least when $\Box$ is interpreted as logical necessity in the framework of logical atomism). The above clause, in other words, but where FM_{CN} is replaced by FM, can be directly inserted into the inductive definition of truth in a truth-value assignment, thereby yielding an inductive definition of truth in a possible world that is applicable to modal formulas as well. That is, in the inductive definition of '$\models_t$' given above, we can add the following new inductive clause:

(4) $\models_t \Box\varphi$ *iff for all* $t' \in V$, $\models_{t'} \varphi$.

The definition of logical truth as truth in all logically possible worlds can now be extended to all formulas in FM, and not just those in FM_{CN}.

Definition 9 *If* $\varphi \in FM$, *then* φ *is* L-true *iff for all* $t \in V$, $\models_t \varphi$.

In regard now to the completeness problem as to which propositional modal calculus has all and only the logical truths as its theorems, we observe first that such a system must contain at least the system $S5$. The axiom schemes of $S5$ are as follows.

1. If φ is tautologous, then $\vdash_{S5} \varphi$;

2. $\vdash_{S5} \Box\varphi \to \varphi$;

3. $\vdash_{S5} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;

4. $\vdash_{S5} \neg\Box\varphi \to \Box\neg\Box\varphi$.

The inference rules of $S5$ are modus ponens (MP) and the rule of necessitation (N):

MP: If $\vdash_{S5} \varphi$ and $\vdash_{S5} (\varphi \to \psi)$, then $\vdash_{S5} \psi$.

N: If $\vdash_{S5} \varphi$, then $\vdash_{S5} \Box\varphi$.

Theorem 10 *If $\vdash_{S5} \varphi$, then φ is L-true.*

Proof. If φ is tautologous, then, as already proved above, φ is L-true. For axiom 2, suppose $t \in V$ and $\models_t \Box\varphi$; then for $t' \in V$, $\models_{t'} \varphi$, and hence $\models_t \varphi$, which shows that $\Box\varphi \to \varphi$ is true at every $t \in V$ and hence that it is L-true. The proof for axiom 3 is similar. For axiom 4, suppose $t \in V$ and that $\models_t \neg\Box\varphi$, and suppose also by *reductio* that $\not\models_t \Box\neg\Box\varphi$. Then, by the truth clause for $\Box$, there must be some $t' \in V$ such that $\not\models_{t'} \neg\Box\varphi$, and hence that $\models_{t'} \Box\varphi$, which means that for all $t'' \in V$, $\models_{t''} \varphi$; but that is impossible because, by hypothesis, there is a $t'' \in V$ such that $\not\models_{t''} \varphi$. Therefore, if $\models_t \neg\Box\varphi$, then $\models_t \Box\neg\Box\varphi$, which shows that axiom 4 is L-true. Finally, it is clear that if φ and $\varphi \to \psi$ are L-true, then so is ψ as well as $\Box\varphi$, which shows that L-truth is preserved under the inference rules MP and N. $\blacksquare$

The question now is does the converse of this theorem also hold? The answer, as the following lemmas indicates, is negative, i.e., not every logical truth as defined above is a theorem of $S5$. First, let us note that the rule of uniform substitution is valid in $S5$, i.e., if $\vdash_{S5} \varphi$, then $\vdash_{S5} \varphi(P_n/\psi)$, where $\varphi(P_n/\psi)$ is the result of uniformly substituting ψ for P_n in φ.

Lemma 11 *US: If $\vdash_{S5} \varphi$, the $\vdash_{S5} \varphi(P_n/\psi)$.*

Proof. Note first that if φ is a tautology, the so is $\varphi(P_n/\psi)$. Also, if φ is an instance of axiom schemes 2, 3, or 4, then $\varphi(P_n/\psi)$ is also an instance of the same axiom schema. Finally, we note also that the inference rules MP and N preserve provability in $S5$ under US, from all of which the validity of US in $S5$ follows. $\blacksquare$

In the next lemma we note that a certain type of formula, namely, $\neg\Box\varphi$, where φ is modal free but not tautologous, is L-true. This is as it should be if

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logical necessity is the counterpart in modal logic of logical truth in semantics. That is, if φ is modal free and not tautologous, then φ is not L-true, and therefore it should be that φ is not logically necessary, which is in fact the case in our semantics.

Lemma 12 *If φ is a modal free and not tautologous, then $\neg\Box\varphi$ is L-true.*

Proof. If φ is a modal free and not tautologous, then there must be some $t \in V$ such that $\nvDash_t \varphi$, and hence by the truth clause for $\Box$, $\nvDash_{t'} \Box\varphi$, for all $t' \in V$, which means that $\vDash_{t'} \neg\Box\varphi$, for all $t' \in V$, and hence that $\neg\Box\varphi$ is L-true. ∎

Finally, we note that even though P_n is modal free and not tautologous, for all $n \in \omega$, and hence by the previous lemma that $\neg\Box P_n$ is L-true, nevertheless $\neg\Box P_n$ is not a theorem of $S5$. In other words, not all logical truths are theorems of $S5$.

Lemma 13 *(1) $\neg\Box P_n$ is not a theorem of $S5$, i.e., $\nvdash_{S5} \neg\Box P_n$; and (therefore) (2) not every L-true formula is a theorem of $S5$.*

Proof. If $\neg\Box P_n$ were a theorem of $S5$, then, by the rule of uniform substitution, US, $\neg\Box P_n(P_n/\varphi\vee\neg\varphi)$ would also be a theorem of $S5$, i.e., then $\vdash_{S5} \neg\Box(\varphi\vee\neg\varphi)$, for all formulas φ. But $\vdash_{S5} \varphi\vee\neg\varphi$, and therefore, by the rule N, $\vdash_{S5} \Box(\varphi\vee\neg\varphi)$, which would mean that $S5$ is inconsistent. But $S5$ is consistent, not inconsistent, which, by *reductio*, shows that $\nvdash_{S5} \neg\Box P_n$. ∎

What the above proof shows, incidentally, is that logical truth is not preserved under the rule US of uniform substitution. In particular, note that whereas $\neg\Box P_n$ is L-true, the result of substituting $(\varphi\vee\neg\varphi)$ for P_n in $\neg\Box P_n$, namely $\neg\Box(\varphi\vee\neg\varphi)$, is not L-true—and, in fact, it is L-false. Uniform substitution can take us not only from logical truths to nonlogical truths, in other words, but to logical falsehoods as well. The reason is that unlike sentence letters, not all formulas—e.g, $(\varphi\vee\neg\varphi)$—represent an atomic situation in logical atomism.

Before concluding this section, there is one useful fact about modally-closed formulas in $S5$, namely that they are provably equivalent to their necessitations. We define modally-closed formulas as follows, and then prove this result for $S5$. We assume in the proof that the following theses are well-known theorems of $S5$. We state the theses here as lemmas.

Definition 14 *φ is a modally closed formula if, and only if, $\varphi \in FM$ and every occurrence of a sentence letter in φ occurs within the scope of an occurrence of $\Box$.*

Lemma 15 *(The Brouwerische thesis) For all $\varphi \in FM$, $\vdash_{S5} \varphi \rightarrow \Box\Diamond\varphi$.*

Lemma 16 *For all $\varphi, \psi \in FM$, $\vdash_{S5} (\Diamond\varphi \rightarrow \Box\psi) \rightarrow \Box(\varphi \rightarrow \psi)$.*

Lemma 17 *(The S4 axiom) For all $\varphi \in FM$, $\vdash_{S5} \Box\varphi \rightarrow \Box\Box\varphi$.*

Theorem 18 *If φ is a modally closed formula, then $\vdash_{S5} (\varphi \leftrightarrow \Box\varphi)$.*

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Proof. Note that because $\Box\varphi \to \varphi$ is an axiom schema of $S5$, it suffices to show by induction on FM that if φ is a modally closed, then $\vdash_{S5} \varphi \to \Box\varphi$. Accordingly, let $\Gamma = \{\varphi \in FM : \text{if } \varphi \text{ is modally closed, then } \vdash_{S5} \varphi \to \Box\varphi\}$. Now. because P_n is not modally closed, for all $n \in \omega$, it follows vacuously that $P_n \in \Gamma$. Suppose then that $\varphi \in \Gamma$ and show that $\neg\varphi \in \Gamma$. Assume, accordingly that $\neg\varphi$ is modally closed, from which it follows that φ must also be modally closed, and hence, by the inductive hypothesis, $\vdash_{S5} \varphi \to \Box\varphi$, and therefore $\vdash_{S5} \varphi \to \Box\neg\neg\varphi$ as well, from which, by truth-functional logic, it follows that $\vdash_{S5} \Diamond\neg\varphi \to \neg\varphi$. But then, by the inference rule N and axiom schema 3, $\vdash_{S5} \Box\Diamond\neg\varphi \to \Box\neg\varphi$. But, by the above (Brouwerische) lemma, $\vdash_{S5} \neg\varphi \to \Box\Diamond\neg\varphi$, from which it follows that $\vdash_{S5} \neg\varphi \to \Box\neg\varphi$, and hence that $\neg\varphi \in \Gamma$.

Suppose now that $\varphi, \psi \in \Gamma$, and show that $(\varphi \to \psi) \in \Gamma$. Assume that $(\varphi \to \psi)$ is modally closed; then so are φ and ψ, and therefore, so is $\neg\varphi$, and hence, by the inductive hypothesis and because $\neg\varphi \in \Gamma$, $\vdash_{S5} \neg\varphi \to \Box\neg\varphi$ and $\vdash_{S5} \psi \to \Box\psi$. Now, by the above lemma, $\vdash_{S5} (\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$, and hence, by truth-functional logic, $\vdash_{S5} (\Box\neg\varphi \vee \Box\psi) \to \Box(\varphi \to \psi)$; and therefore $\vdash_{S5} (\neg\varphi \vee \psi) \to \Box(\varphi \to \psi)$, i.e., $\vdash_{S5} (\varphi \to \psi) \to \Box(\varphi \to \psi)$, from it follows that $(\varphi \to \psi) \in \Gamma$.

Finally, assume $\varphi \in \Gamma$ and show that $\Box\varphi \in \Gamma$. But by the above $(S4)$ lemma, $\vdash_{S5} \Box\varphi \to \Box\Box\varphi$, from which it follows that $\Box\varphi \in \Gamma$. We conclude by the induction principle that $FM \subseteq \Gamma$. $\blacksquare$

4 A Modal Logic for Logical Atomism

If $S5$ is not complete with respect to logical truth, what, if any, modal calculus is? As we will see, the system described in this section contains $S5$ and yields a strong completeness theorem for logical necessity as explicated above. Because this calculus can be taken to represent logical atomism (on the sentential level of analysis), we will refer to it as L_{at}.[8]

The inference rules of L_{at} are MP and N as in $S5$, and the axioms are as follows:

1. If φ is tautologous, then $\vdash_{L_{at}} \varphi$;

2. $\vdash_{L_{at}} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$; and

3. If φ is modal free and not tautologous, then $\vdash_{L_{at}} \neg\Box\varphi$.

It is of course clear from theorem 10 and lemma 12 that every theorem of L_{at} is L-true, and hence, because no formula can be both true and false in the same truth-value assignment, that L_{at} is consistent.

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[8]The axioms of L_{at} are simpler than the system described for this purpose in chapter 6 of Cocchiarella 1987 (which was originally published in 1974). The simplification was given in Carroll 1978.

Theorem 19 *If $\vdash_{L_{at}} \varphi$, then φ is L-true.*

The next theorem indicates that in logical atomism no new "facts" of the world are described by means of modal formulas over and above those that are described by modal-free formulas—because, according to that lemma, whatever can be described by means of a modal formula can also be described by a provably equivalent modal-free formula. This is as it should be in logical atomism where all facts, i.e., states of affairs that obtain, are ultimately reducible to (or analyzable in terms of) atomic facts.

Theorem 20 *For all $\varphi \in FM$, there is a modal free formula ψ such that $\vdash_{L_{at}} (\varphi \leftrightarrow \psi)$.*

Proof. Let $\Gamma = \{\varphi \in FM : $ for some modal-free $\psi \in FM_{CN}, \vdash_{L_{at}} (\varphi \leftrightarrow \psi)\}$. It suffices to show by induction on FM that $FM \subseteq \Gamma$. Suppose first that $n \in \omega$. Then, because $\vdash_{L_{at}} (P_n \leftrightarrow P_n)$, $P_n \in \Gamma$. Suppose now that $\varphi \in \Gamma$ and show $\neg\varphi \in \Gamma$. By assumption, for some $\psi \in FM_{CN}, \vdash_{L_{at}} (\varphi \leftrightarrow \psi)$, and therefore, by truth-functional logic, $\vdash_{L_{at}} (\neg\varphi \leftrightarrow \neg\psi)$. But $\neg\psi \in FM_{CN}$, so therefore $\neg\varphi \in \Gamma$. Suppose now that $\varphi, \chi \in \Gamma$ and show $(\varphi \rightarrow \chi) \in \Gamma$. By assumption, $\vdash_{L_{at}} (\varphi \leftrightarrow \psi)$, for some $\psi \in FM_{CN}$, and $\vdash_{L_{at}} (\chi \leftrightarrow \psi')$, for some $\psi' \in FM_{CN}$; and therefore, by truth-functional logic, $\vdash_{L_{at}} (\varphi \rightarrow \chi) \leftrightarrow (\psi \rightarrow \psi')$. But $(\psi \rightarrow \psi') \in FM_{CN}$, so therefore $(\varphi \rightarrow \chi) \in \Gamma$. *Finally*, suppose $\varphi \in \Gamma$ and show $\Box\varphi \in \Gamma$. By assumption, $\vdash_{L_{at}} (\varphi \leftrightarrow \psi)$, for some $\psi \in FM_{CN}$; therefore, by the inference rule N, axiom schema 2, and truth-functional logic, $\vdash_{L_{at}} (\Box\varphi \leftrightarrow \Box\psi)$. We consider two subcases depending on whether or not ψ, which is modal-free, is tautologous or not. Suppose, first, that ψ is tautologous. Then, $\vdash_{L_{at}} \psi$, which means, by the rule N, that $\vdash_{L_{at}} \Box\psi$, and therefore, by truth-functional logic, $\vdash_{L_{at}} \Box\varphi$. Consequently, again by truth-functional logic, $\vdash_{L_{at}} (\Box\varphi \leftrightarrow [P_n \vee \neg P_n])$; from which it follows, because $(P_n \vee \neg P_n) \in FM_{CN}$, that $\Box\varphi \in \Gamma$. Suppose now that ψ is not tautologous. Then, by definition, $\neg\Box\psi$ is an axiom of L_{at}, and therefore $\vdash_{L_{at}} \neg\Box\psi$, from which, by truth-functional logic, it follows that $\vdash_{L_{at}} \neg\Box\varphi$. Consequently, by truth-functional logic, $\vdash_{L_{at}} (\Box\varphi \leftrightarrow \neg[P_n \vee \neg P_n])$, where $[P_n \vee \neg P_n] \in FM_{CN}$; and so, in this case as well, $\Box\varphi \in \Gamma$. That is, whether ψ is tautologous or not, $\Box\varphi \in \Gamma$. ∎

The next theorem is both useful for what follows and appropriate in regard to logical necessity. It says, in effect, that if a formula φ is not provable, then the logical possibility of its being false is provable, i.e., then $\Diamond\neg\varphi$ is provable.

Theorem 21 *For all $\varphi \in FM$, either $\vdash_{L_{at}} \varphi$ or $\vdash_{L_{at}} \neg\Box\varphi$.*

Proof. By the previous theorem, for some modal-free $\psi \in FM_{CN}, \vdash_{L_{at}} (\varphi \leftrightarrow \psi)$, and therefore, by the rule N, axiom schema 2 and truth-functional logic, $\vdash_{L_{at}} (\Box\varphi \leftrightarrow \Box\psi)$, from which, by truth-functional logic, it follows that $\vdash_{L_{at}} (\neg\Box\varphi \leftrightarrow \neg\Box\psi)$. If ψ is tautologous, then $\vdash_{L_{at}} \psi$, and therefore, by truth-functional logic, $\vdash_{L_{at}} \varphi$. On the other hand, if ψ is not tautologous, then $\vdash_{L_{at}} \neg\Box\psi$, and therefore, by truth-functional logic, $\vdash_{L_{at}} \neg\Box\varphi$. Therefore, either $\vdash_{L_{at}} \varphi$ or $\vdash_{L_{at}} \neg\Box\varphi$. ∎

Theorem 22 *(1) $\vdash_{L_{at}} \Box\varphi \to \varphi$; and (2) $\vdash_{L_{at}} \Diamond\varphi \to \Box\Diamond\varphi$.*

Proof. For (1), we have, by the previous theorem, either $\vdash_{L_{at}} \varphi$ or $\vdash_{L_{at}}\neg\Box\varphi$. But, by truth-functional logic, $\vdash_{L_{at}} \varphi \to (\Box\varphi \to \varphi)$ and $\vdash_{L_{at}}\neg\Box\varphi \to (\Box\varphi \to \varphi)$, and therefore, in either case, by the MP rule, $\vdash_{L_{at}} \Box\varphi \to \varphi$.

For (2), we also have, by the previous theorem, either $\vdash_{L_{at}} \neg\varphi$ or $\vdash_{L_{at}}\neg\Box\neg\varphi$; and therefore, by the rule N and the definition of $\Diamond$, either $\vdash_{L_{at}} \Box\neg\varphi$ or $\vdash_{L_{at}} \Diamond\varphi$, i.e., either $\vdash_{L_{at}}\neg\Diamond\varphi$ or $\vdash_{L_{at}} \Diamond\varphi$, and therefore, again by the rule N, either $\vdash_{L_{at}}\neg\Diamond\varphi$ or $\vdash_{L_{at}} \Box\Diamond\varphi$. But, by truth-functional logic, $\vdash_{L_{at}}\neg\Diamond\varphi \to (\Diamond\varphi \to \Box\Diamond\varphi)$ and $\vdash_{L_{at}} \Box\Diamond\varphi \to (\Diamond\varphi \to \Box\Diamond\varphi)$; and so in either case $\vdash_{L_{at}} \Diamond\varphi \to \Box\Diamond\varphi$, which completes the proof of (2). ∎

We note that $\Diamond\varphi \to \Box\Diamond\varphi$ is equivalent to $\neg\Box\varphi \to \Box\neg\Box\varphi$, which means that every instance of axiom schemes 2 and 4 of $S5$ are theorems of L_{at}, and that L_{at} is an extension of $S5$. Of course, because the rule US of uniform substitution is not valid in L_{at}, this means that L_{at} is a nonclassical system, and hence a nonclassical extension of $S5$.

Theorem 23 *L_{at} is a nonclassical extension of $S5$.*

Now if the necessity sign really does represent logical necessity, then it would seem that any modally closed formula should be either L-true or L-false (i.e., its negation should then be L-true). Accordingly, if L_{at} does yield a complete representation of logical necessity (as understood in logical atomism), then every modally closed formula should be either provable or refutable in L_{at}. This in fact is the case, as is indicated in the following theorem.

Theorem 24 *If φ is modally closed, then either $\vdash_{L_{at}} \varphi$ or $\vdash_{L_{at}}\neg\varphi$.*

Proof. Assume the hypothesis. Suppose φ is not provable in L_{at}, i.e., suppose $\nvdash_{L_{at}} \varphi$. Then, by theorem 21, $\vdash_{L_{at}}\neg\Box\varphi$. But, by assumption φ is modally closed, so by theorem 18, $\vdash_{S5} (\varphi \leftrightarrow \Box\varphi)$, and hence, because L_{at} contains $S5$, $\vdash_{L_{at}} (\varphi \leftrightarrow \Box\varphi)$. Therefore, by truth-functional logic, $\vdash_{L_{at}}\neg\varphi$. ∎

5　Maximally Consistent Sets and Strong Completeness

In addition to provability in L_{at} there is also the relation of derivability between a set of formula and a formula, which we can define in terms of provability as follows.

Definition 25 *If $K \cup \{\varphi\} \subseteq FM$, then K yields φ in L_{at}, in symbols $K \vdash_{L_{at}} \varphi$, if, and only if, for some $n \in \omega$ and $\psi_0, ..., \psi_{n-1} \in K$, $\vdash_{L_{at}} (\psi_0 \wedge ... \wedge \psi_{n-1} \to \varphi)$.*[9]

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[9]Note that, by convention, if $n = 0$, then $(\psi_0 \wedge ... \wedge \psi_{n-1} \to \varphi)$ is just φ.

The next two definitions make explicit the notions of consistency and maximal consistency in L_{at}. Maximal consistency amounts to a complete description of a possible world, and in that sense maximally consistent sets are the syntactical correlates of possible worlds. Representing possible worlds in terms of such syntactical correlates enables us to prove the strong completeness of L_{at}.

Definition 26 *If $K \subseteq FM$, then K is consistent in L_{at} if, and only if, there is no formula $\varphi \in FM$ such that $K \vdash_{L_{at}} \varphi$ and $K \vdash_{L_{at}} \neg\varphi$.*

Definition 27 *If $K \subseteq FM$, then K is maximally consistent in L_{at}, in symbols $K \in MC_{L_{at}}$, if, and only if, K is consistent in L_{at} and for all $\varphi \in FM$, either $\varphi \in K$ or $K \cup \{\varphi\}$ is inconsistent in L_{at}.*

The following lemma states three well-known and obvious properties of maximal consistency that are easily proved.

Lemma 28 *If $K \cup \{\varphi\} \subseteq FM$ and $K \in MC_{L_{at}}$, then:*
(1) $\varphi \in K$ if, and only if, $\neg\varphi \notin K$;
(2) $(\varphi \to \psi) \in K$ if, and only if, either $\varphi \notin K$ or $\psi \in K$; and
(3) if $\vdash_{L_{at}} \varphi$, then $\varphi \in K$.

Also, we will find the following theorem useful in proving strong completeness. We assume for its proof Lindenbaum's lemma that every consistent set of formulas can be extended to a maximally consistent set. The maximally consistent set in Lindenbaum's lemma is proved to exist by a well-known type of construction that we won't go into here.[10]

Theorem 29 *If $K \cup \{\phi\} \subseteq FM$, then $K \vdash_{\Sigma} \phi$ if, and only if, for all $\Gamma \in MC_{L_{at}}$, if $K \subseteq \Gamma$, then $\phi \in \Gamma$.*

Proof. Assume the hypothesis. We first prove the left-to-right direction and assume $K \vdash_{L_{at}} \varphi$, $\Gamma \in MC_{L_{at}}$, and $K \subseteq \Gamma$, and show that $\varphi \in \Gamma$. We note that, by hypothesis and definition, $\Gamma \vdash_{L_{at}} \varphi$; and, therefore, because Γ is (maximally) consistent in L_{at}, $\Gamma \nvdash_{\Sigma} \neg\varphi$. Now, by lemma 28, $\neg\varphi \in \Gamma$ iff $\varphi \notin \Gamma$; but if $\neg\varphi \in \Gamma$, then, by definition (and the fact that $\vdash_{L_{at}} (\neg\varphi \to \neg\varphi)$), $\Gamma \vdash_{\Sigma} \neg\varphi$, which, as already noted, is not so, which means that $\neg\varphi \notin \Gamma$, and hence $\varphi \in \Gamma$, which was to be shown.

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[10]A brief sketch of Lindenbaum's lemma is as follows. Suppose we have a consistent set $K \subseteq FM$, and that $\varphi_1, ..., \varphi_n$ is an enumeration of FM. Then we recursively define a function Γ as follows:

1. $\Gamma_0 = K$

2. $\Gamma_{n+1} = \begin{cases} \Gamma_n \text{ if } \Gamma_n \vdash_{L_{at}} \neg\varphi_{n+1} \\ \Gamma_n \cup \{\varphi_{n+1}\} \text{ otherwise} \end{cases}$.

We then show Γ_n is consistent for all $n \in \omega$, from which it follows that $\Gamma^* = \cup_{n \in \omega} \Gamma_n$ is also consistent, and yet $K \subseteq \Gamma^* \in MC_{L_{at}}$.

For the converse direction, assume now instead that for all $\Gamma \in MC_{L_{at}}$, if $K \subseteq \Gamma$, then $\varphi \in \Gamma$, and show that $K \vdash_\Sigma \varphi$. Assume, by *reductio*, that $K \nvdash_\Sigma \varphi$. Then $K \cup \{\neg\varphi\}$ is consistent in L_{at}, and therefore, by Lindenbaum's lemma, there is a $\Gamma \in MC_{L_{at}}$ such that $K \cup \{\neg\varphi\} \subseteq \Gamma$. But then $K \subseteq \Gamma$, and so therefore, by assumption, $\varphi \in \Gamma$. But we also have $\neg\varphi \in \Gamma$, which means that both $\Gamma \vdash_{L_{at}} \varphi$ and $\Gamma \vdash_{L_{at}} \neg\varphi$, i.e., that Γ is not consistent in L_{at}, which is impossible because, by assumption, Γ is (maximally) consistent in L_{at}. $\blacksquare$

In general, if K is a maximally consistent set of formulas, i.e., $K \in MC_{L_{at}}$, then there is exactly one truth-value assignment $t \in V$ such that for all $n \in \omega$, $t(P_n) = 1$ iff $P_n \in K$. We shall use 't_K' to represent this unique truth-value assignment. In L_{at}, as our next theorem indicates, membership in a maximally consistent set K amounts—for all formulas, and not just sentence letters—to truth in the possible world represented by t_K.

Definition 30 *If $K \in MC_{L_{at}}$, then $t_K =_{df}$ the $t \in V$ such that for all $n \in \omega$, $t(P_n) = 1$ iff $P_n \in K$.*

Theorem 31 *If $K \in MC_{L_{at}}$, then $\varphi \in K$ iff $\models_{t_K} \varphi$.*

Proof. Assume the hypothesis and let $\Gamma = \{\varphi \in FM : \varphi \in K$ iff $\models_{t_K} \varphi\}$. It suffices to show by induction on FM that $FM \subseteq \Gamma$. There are four cases to consider. Case 1: by definition, for all $n \in \omega$, $P_n \in K$ iff $\models_{t_K} P_n$, so therefore $P_n \in \Gamma$. Case 2: Assume $\varphi \in \Gamma$, and show that $\neg\varphi \in \Gamma$. Then, by the inductive hypothesis, $\varphi \in K$ iff $\models_{t_K} \varphi$, and therefore $\varphi \notin K$ iff $\nvDash_{t_K} \varphi$, and hence, by lemma 28, $\neg\varphi \in K$ iff $\models_{t_K} \neg\varphi$, which shows that $\neg\varphi \in \Gamma$. Case 3: Assume $\varphi, \psi \in \Gamma$, and show that $(\varphi \to \psi) \in \Gamma$. Then, by case 2, $\neg\varphi \in \Gamma$, i.e., $\neg\varphi \in K$ iff $\models_{t_K} \neg\varphi$, and by the inductive hypothesis, $\psi \in K$ iff $\models_{t_K} \psi$. But, by the above lemma 28, $(\varphi \to \psi) \in K$ iff $\varphi \notin K$ or $\psi \in K$, i.e., iff $\nvDash_{t_K} \varphi$ or $\models_{t_K} \psi$, and hence, iff $\models_{t_K} (\varphi \to \psi)$, which shows that $(\varphi \to \psi) \in \Gamma$. *Case 4*: Suppose $\varphi \in \Gamma$ and show $\Box\varphi \in \Gamma$. Now, by theorem 24, $\vdash_{L_{at}} \Box\varphi$ or $\vdash_{L_{at}} \neg\Box\varphi$. Suppose first that $\vdash_{L_{at}} \Box\varphi$. Then, by lemma 28, $\Box\varphi \in K$. Also, by theorem 19, $\Box\varphi$ is L-true, which means, by definition, that for all $t \in V$, $\models_t \Box\varphi$; hence, because $t_K \in V$, $\models_{t_K} \Box\varphi$. It follows, accordingly, that $\Box\varphi \in K$ iff $\models_{t_K} \Box\varphi$. Suppose now that $\vdash_{L_{at}} \neg\Box\varphi$. Then, because $K \in MC_{L_{at}}$, $\neg\Box\varphi \in K$; and therefore, by lemma 28, $\Box\varphi \notin K$. Also, by theorem 19, $\neg\Box\varphi$ is L-true; hence, by definition, for all $t \in V$, $\models_t \neg\Box\varphi$. But $t_K \in V$, and so $\models_{t_K} \neg\Box\varphi$, and therefore $\nvDash_{t_K} \Box\varphi$. Accordingly, $\Box\varphi \notin K$ iff $\nvDash_{t_K} \Box\varphi$; and therefore, $\Box\varphi \in K$ iff $\models_{t_K} \Box\varphi$. Thus in case either $\vdash_{L_{at}} \Box\varphi$ or $\vdash_{L_{at}} \neg\Box\varphi$, $\Box\varphi \in \Gamma$. It follows by the induction principle that $FM \subseteq \Gamma$. $\blacksquare$

In logical atomism, as we have already noted, logically possible worlds are completely determined by the atomic states of affairs that obtain in those worlds. This means in particular that no "new" facts are represented by conditional formulas or the negations of formulas other than sentence letters. It also means, as noted above, that there are no modal facts, i.e., facts represented by modal formulas that are not reducible to the atomic facts represented by sentence

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letters. In other words, in logical atomism, worlds that are indiscernible in their atomic facts are indiscernible in their modal facts as well.

Any calculus that purports to represent logical atomism, accordingly, must be such that maximally consistent sets of formulas of that system will be identical if, and only if, they coincide on the atomic sentences in those sets, i.e., if, and only if, they determine the same truth-value assignment (as a semantic representation of a logically possible world). In terms of this criterion of adequacy, we justify our claim in the following lemma, which is an immediate consequence of theorem 31, that L_{at} is an adequate representation of logical atomism.

Lemma 32 *If K, $K' \in MC_{L_{at}}$ and $t_K = t_{K'}$, then $K = K'$.*

We are now ready to prove the completeness theorem for L_{at}. In doing so we first introduce the notion of logical implication, or, for brevity, *L-implication*, that corresponds to derivability the way that provability in L_{at} corresponds to *L*-truth as defined above (or so we will prove). Intuitively, the idea is that a set of premises logically implies a conclusion φ if, and only if, φ is true in every logically possible world in which all of the premises are true. In theorem 34, we show that logical implication (as explicated here) coincides with derivability in L_{at}, which is our strong completeness theorem. An immediate corollary is that logical truth (as explicated here) coincides with provability in L_{at}, which is the same as derivability from zero many premises, i.e., from the empty set.

Definition 33 *If $\Gamma \cup \{\varphi\} \subseteq FM$, then Γ L-implies φ iff for all $t \in V$, if $\models_t \psi$, for all $\psi \in \Gamma$, then $\models_t \varphi$*

Theorem 34 *(Strong Completeness)*: $\Gamma \vdash_{L_{at}} \varphi$ *iff* Γ *L-implies* φ.

Proof. Suppose first that $\Gamma \vdash_{L_{at}} \varphi$ and show that Γ *L*-implies φ. Then, by definition, $\vdash_{L_{at}} (\psi_0 \wedge ... \wedge \psi_{n-1} \to \varphi)$, for some $\psi_0, ..., \psi_{n-1} \in \Gamma$; and hence, by theorem 19, $(\psi_0 \wedge ... \wedge \psi_{n-1} \to \varphi)$ is *L*-true. Suppose now that $t \in V$ and that for all $\psi \in \Gamma$, $\models_t \psi$. Then, by assumption, $\models_t (\psi_0 \wedge ... \wedge \psi_{n-1})$ and, by definition of *L*-truth, $\models_t (\psi_0 \wedge ... \wedge \psi_{n-1} \to \varphi)$, from which it follows that $\models_t \varphi$; and hence that Γ *L*-implies φ.

For the converse direction, suppose now that Γ *L*-implies φ and show $\Gamma \vdash_{L_{at}} \varphi$. Then, by theorem 29, it is sufficient to show that for all $K \in MC_{L_{at}}$, if $\Gamma \subseteq K$, then $\varphi \in K$. Suppose, accordingly, that $K \in MC_{L_{at}}$ and that $\Gamma \subseteq K$. By theorem 31, for all ψ, $\psi \in K$ iff $\models_{t_K} \psi$. Therefore, for all $\psi \in \Gamma$, $\models_{t_K} \psi$; hence, by assumption and definition of *L*-implication, $\models_{t_K} \varphi$, from which it follows, by theorem 31, that $\varphi \in K$. ∎

Corollary 35 *(Weak Completeness)*: $\vdash_{L_{at}} \varphi$ *iff φ is L-true.*

In addition to the syntactical notion of consistency in L_{at}, we also have a semantical notion. That is, a set of formulas is *semantically consistent* if, and only if, there is some logically possible world in which every formula in the set is

true. Theorem 37 below, which indicates that the syntactic and semantic notions of consistency coincide, amounts to another version of the strong completeness theorem for L_{at}.

Definition 36 *K is semantically consistent iff for some $t \in V$, $\models_t \psi$, for all $\psi \in K$.*

Theorem 37 *K is semantically consistent iff K is (syntactically) consistent in L_{at}.*

Proof. Suppose first that K is semantically consistent, and, by *reductio*, that K is not (syntactically) consistent in L_{at}. Then, for some $t \in V$, $\models_t \psi$, for all $\psi \in K$, and yet for some $\psi \in FM$, $K \vdash_{L_{at}} \psi$ and $K \vdash_{L_{at}} \neg\psi$. But then, by strong completeness, $K \models_t \psi$ and $K \models_t \neg\psi$, which is impossible, because then $\models_t \psi$ and $\models_t \neg\psi$. For the opposite direction, assume K is (syntactically) consistent in L_{at}. Then, by Lindenbaum's lemma, there is a maximally consistent set $\Gamma \in MC_{L_{at}}$ such that $K \subseteq \Gamma$. But then, by theorem 31, $\vdash_{t_\Gamma} \psi$ for all $\psi \in \Gamma$, and therefore $\vdash_{t_\Gamma} \psi$ for all $\psi \in K$, which shows that K is semantically consistent. ∎

This completes our account of the semantics for logical necessity as based on Carnap's criterion of adequacy and of the modal propositional calculus L_{at} that is complete with respect to that semantics. It is instructive to consider how this semantics differs from one similar to it but which is complete for $S5$. The fundamental difference, it turns out, is a secondary reading, or "cut down", on the notion all possible worlds, and what comes with this secondary reading are the possibility of modal facts as part of the world over and above the atomic facts that make it up in logical atomism.

6 A Semantics for S5: All Possible Worlds "Cut Down"

Our reformulation of Carnap's criterion of adequacy for logical necessity as a truth-condition for formulas of the form $\Box\varphi$ construes necessity as a universal quantifier over all logically possible worlds—where each logically possible world is represented by a truth-value assignment, i.e., by a specification of all of the atomic states of affairs that obtain in that world. On this interpretation, as we saw above, there are more logical truths than there are theorems of $S5$.

It is possible to give a restricted, or secondary, interpretation of the notion of *all* possible worlds, however, under which the valid formulas are none other than the theorems of $S5$—i.e., an interpretation with respect to which we can obtain a completeness theorem for $S5$. The restricted, or secondary, interpretation for necessity is similar to the restricted interpretation for quantification over arbitrary properties, or classes, in second-order predicate logic, where the interpretation involves structures called "nonstandard" models. The idea of this interpretation is to deal not necessarily with the whole of logical space, i.e., with

all logically possible worlds (as explicated in logical atomism), but with arbitrary regions of logical space, by which we mean arbitrary nonempty classes of possible worlds. The new interpretation of necessity then is not as a quantifier over all logically possible worlds but rather with all the possible worlds (truth-value assignments) in a given region of logical space, i.e., in a given nonempty class of possible worlds. For this reason, the notion of truth (or falsity) is no longer simply truth (or falsity) in a given logically possible world, but truth (or falsity) in a possible world *relative to* a given class of possible worlds (or region of logical space), a notion that allows for the importation of material content into the meaning of necessity, and hence something other than logical necessity.

Definition 38 *If $T \subseteq V$ and $t \in T$, then:*
(1) $\models_t^T P_n$ iff $t(P_n) = 1$;
(2) $\models_t^T \neg\varphi$ iff $\not\models_t^T \varphi$;
(3) $\models_t^T (\varphi \to \psi)$ iff either $\not\models_t^T \varphi$ or $\models_t^T \psi$; and
(4) $\models_t^T \Box\varphi$ iff for all $t' \in T$, $\models_{t'}^T \varphi$.

Note: We read '$\models_t^T \varphi$' as 'φ *is true in (region) T at (world) t*'.

One invariance condition we can now define is truth at all worlds in a region of logical space, i.e., at all worlds in the class of worlds making up that region. If T is a nonempty subset of V, then invariant truth at all of the worlds in T will be called *T-validity*.

Definition 39 *If $T \subseteq V$ and $T \neq 0$, then φ is T-valid iff for all $t \in T$, $\models_t^T \varphi$.*

Of course, logical truth, i.e., truth in all logically possible worlds of logical space, is the most general invariance condition of truth, which means that if φ is L-true, then φ is T-valid for all $T \subseteq V$.

Lemma 40 *If φ is L-true, then φ is T-valid for all nonempty $T \subseteq V$.*

After logical truth as invariant truth in all logically possible worlds, the next most general notion of invariant truth is T-validity for all regions T of logical space, i.e., truth at every world in every region of logical space. We will call this notion *validity simpliciter*. For convenience, we will also speak of valid implication in this restricted sense between a set of premises and a conclusion as *v-implication*.

Definition 41 *If $K \cup \{\varphi\} \subseteq FM$, then:*
(1) φ is valid iff for all nonempty $T \subseteq V$, φ is T-valid; and
(2) K v-implies φ iff for all nonempty $T \subseteq V$ and all $t \in T$, if $\models_t^T \psi$, for all $\psi \in K$, then $\models_t^T \varphi$.

It follows from the previous lemma that if φ is L-true, then φ is valid *simpliciter*. But as already noted in theorem 10, all theorems of $S5$ are L-true; and so therefore all theorems of $S5$ are valid *simpliciter*.

Lemma 42 *If $\vdash_{S5} \varphi$, then φ is valid.*

Derivability in $S5$ is defined in an entirely similar way as derivability in L_{at}, except of course relative to the axioms of $S5$ as opposed to those of L_{at}.

Definition 43 *If $K \cup \{\varphi\} \subseteq FM$, then $K \vdash_{S5} \varphi$ if,and only if, for some $n \in \omega$ and some $\psi_0, ..., \psi_{n-1} \in K, \vdash_{S5} \psi_0 \wedge ... \wedge \psi_{n-1} \to \varphi$.*

As the following theorem indicates, conclusions derivable from premises within $S5$ are v-implied by those premises; that is, $S5$ is *sound* with respect to this interpretation of implication. Because validity is equivalent to v-implication from the empty set of premises, we have the following obvious result.

Theorem 44 *If $\Gamma \vdash_{S5} \varphi$, then Γ v-implies φ.*

Proof. Suppose $\Gamma \vdash_{S5} \varphi$. Then for some $\psi_0, ..., \psi_{n-1} \in \Gamma, \vdash_{S5} \psi_0 \wedge ... \wedge \psi_{n-1} \to \varphi$, and therefore $\psi_0 \wedge ... \wedge \psi_{n-1} \to \varphi$ is valid. But for any $T \subseteq V$ and any $t \in T$ such that for all $\chi \in \Gamma, \models_t^T \chi$, we then have $\models_t^T \psi_i$, for all $i \leq n-1$, and hence by the truth conditions for $\to$, $\models_t^T \varphi$; and therefore, by definition, Γ v-implies φ. $\blacksquare$

7 A Completeness Theorem for $S5$

We saw in regard to the notion of L-truth that if $K, K' \in MC_{L_{at}}$ and $t_K = t_{K'}$, then $K = K'$. By associating each possible world of logical atomism with the maximally L_{at}-consistent class of formulas that represent the facts or states of affairs that obtain in that world, this result indicates that *worlds indiscernible in their atomic (nonmodal) facts are indiscernible in their modal facts as well.* This, it should be emphasized, is a consequence of the semantical clause for necessity that interprets it as a quantifier over *all* logically possible worlds (as explicated in logical atomism).

No such similar result holds in our present secondary semantics for necessity, i.e., the semantics with respect to which $S5$ will be shown to be complete. In particular, where MC_{S5} is the set of all sets of formulas that are maximally consistent in $S5$, we will show that there are $K, K' \in MC_{S5}$ such that $t_K = t_{K'}$, and yet $K \neq K'$. That is, in the worlds represented by $S5$ (or the maximally $S5$-consistent sets of formulas), there are "modal facts" over and above the nonmodal facts that obtain in those worlds. Semantically, the reason for this difference is none other than the fact that necessity is now being interpreted as a restricted quantifier, i.e., as a quantifier not over *all* logically possible worlds, but only over all possible worlds in a region of logical space, i.e., all possible worlds in a given nonempty class of possible worlds.

We are dealing now not with maximally L_{at}-consistent sets of formulas as complete descriptions of possible worlds, it should be emphasized, but with maximally $S5$-consistent sets instead. The definitions of consistency and maximal consistency in $S5$ are assumed to be defined in essentially the same way they are

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defined for L_{at} and that the properties of maximal $S5$-consistent sets of formulas are as they are for maximal L_{at}-consistent sets as described in lemma 28. The question we are concerned with is what conditions of complete (i.e., maximally $S5$-consistent) descriptions of possible worlds suffice for the indiscernibility of those worlds? We answer this question in the following two theorems. In particular, as the second theorem indicates, the possible worlds represented by maximally $S5$-consistent sets of formulas are indiscernible if they contain the same atomic facts and the same necessary facts (and therefore the same possible facts as well).

Theorem 45 *If $\Gamma \in MC_{S5}$, $\Theta = \{K \in MC_{S5} : \text{for all } \varphi, \text{ if } \Box\varphi \in \Gamma, \text{ then } \varphi \in K\}$ and $T = \{t_K : K \in \Theta\}$, then for all $K \in \Theta$:*

(1) $\Box\varphi \in \Gamma$ iff $\Box\varphi \in K$;

(2) if $\varphi \in K$ and $\neg\Box\varphi \in K$, then there is a $K' \in \Theta$ such that $\neg\varphi \in K'$; and

(3) $\varphi \in K$ iff $\models^T_{t_K} \varphi$.

Proof. Assume the hypothesis and that $K \in \Theta$. For (1), suppose first that $\Box\varphi \in \Gamma$; then, because $\vdash_{S5} (\Box\varphi \to \Box\Box\varphi)$ and $\Gamma \in MC_{S5}$, $\Box\Box\varphi \in \Gamma$; and therefore, because $K \in \Theta$, $\Box\varphi \in K$. Suppose, conversely, that $\Box\varphi \in K$ but that $\Box\varphi \notin \Gamma$. Then, because $\Gamma \in MC_{S5}$, $\neg\Box\varphi \in \Gamma$. But $\vdash_{S5} (\neg\Box\varphi \to \Box\neg\Box\varphi)$, and, therefore $\Box\neg\Box\varphi \in \Gamma$, from which it follows by definition of Θ that $\neg\Box\varphi \in K$. That is, K is then inconsistent, which is impossible because $K \in MC_{S5}$. Therefore, $\Box\varphi \in \Gamma$ iff $\Box\varphi \in K$.

For (2), suppose that $\varphi \in K$ and $\neg\Box\varphi \in K$. Let $\Xi = \{\Box\psi : \Box\psi \in K\}$ and show first that $\Xi \cup \{\neg\varphi\}$ is $S5$-consistent. By *reductio*, assume that $\Xi \cup \{\neg\varphi\}$ is not $S5$-consistent, i.e., that $\Xi \cup \{\neg\varphi\} \vdash_{S5} \neg(\chi \to \chi)$, for some χ. Then, by sentential logic and the Deduction Theorem (which is provable for $S5$ in the standard manner), $\vdash_{S5} \Box\psi_0 \wedge ... \wedge \Box\psi_{n-1} \to \varphi$, for some $\Box\psi_0, ..., \Box\psi_{n-1} \in \Xi \subseteq K$. Accordingly, by sentential logic, the distribution of $\Box$ over $\to$, and other well-known properties of $S5$, $\vdash_{S5} \Box\Box\psi_0 \wedge ... \wedge \Box\Box\psi_{n-1} \to \Box\varphi$; and therefore, because $\vdash_{S5} (\Box\psi_i \leftrightarrow \Box\Box\psi_i)$, for all $i < n$, $\vdash_{S5} \Box\psi_0 \wedge ... \wedge \Box\psi_{n-1} \to \Box\varphi$; that is, $\Xi \vdash_{S5} \Box\varphi$. But then, because $\Xi \subseteq K$, $K \vdash_{S5} \Box\varphi$, which is impossible because $\neg\Box\varphi \in K$ and $K \in MC_{S5}$. We conclude, then, that $\Xi \cup \{\neg\varphi\}$ is $S5$-consistent after all. Accordingly, by Lindenbaum's Lemma, there is a set $K' \in MC_{S5}$ such that $\Xi \cup \{\neg\varphi\} \subseteq K'$. But, for all χ, if $\Box\chi \in \Gamma$, then, by (1), because $K \in \Theta$, $\Box\chi \in K$, and therefore, by definition, $\Box\chi \in \Xi \subseteq K'$. But $\vdash_{S5} \Box\chi \to \chi$, and therefore, $\chi \in K'$, from which it follows that $K' \in \Theta$.

For (3), let $\Delta = \{\varphi \in FM : \text{for all } K \in \Theta, \varphi \in K \text{ iff } \models^T_{t_K} \varphi\}$. It suffices to show by induction that $FM \subseteq \Delta$. There are then four cases to consider. Case 1: Suppose $n \in \omega$ and show $P_n \in \Delta$. But by definition of t_K, where $K \in \Theta$, $P_n \in \Delta$. For case 2: Suppose now $\varphi \in \Delta$ and show $\neg\varphi \in \Delta$. But, by the inductive hypothesis, for $K \in \Theta$, $\varphi \in K$ iff $\models^T_{t_K} \varphi$; and therefore $\varphi \notin K$ iff $\nvDash^T_{t_K} \varphi$, from which it follows, because $K \in MC_{S5}$, that $\neg\varphi \in K$ iff $\models^T_{t_K} \neg\varphi$. That is, $\neg\varphi \in \Delta$. For case 3: Suppose $\varphi, \psi \in \Delta$ and show that $(\varphi \to \psi) \in \Delta$. Then, where $K \in \Theta$, by the inductive hypothesis, we have both $(\varphi \in K$ iff $\models^T_{t_K} \varphi)$ and $(\psi \in K$ iff $\models^T_{t_K} \psi)$; and therefore by the truth-clause for $(\varphi \to \psi)$

and the $S5$ analogue of lemma 28, $(\varphi \to \psi) \in K$ iff $\models^T_{t_K} (\varphi \to \psi)$, from which it follows that $(\varphi \to \psi) \in \Delta$. Case 4: Finally, suppose $\varphi \in \Delta$ and show $\Box\varphi \in \Delta$. Assume, accordingly, that $K \in \Theta$ and, for the left-to right direction, that $\Box\varphi \in K$. Then, by (1) above, $\Box\varphi \in \Gamma$. Suppose now that $t \in T$, i.e., that $t = t_{K'}$, for some $K' \in \Theta$, and show $\models^T_t \varphi$. Then, again by (1) above, $\Box\varphi \in K'$; and therefore, because $\vdash_{S5} (\Box\varphi \to \varphi)$, $\varphi \in K'$. Then, by the inductive assumption, $\models^T_{t_{K'}} \varphi$, i.e., $\models^T_t \varphi$; from which we conclude, by the truth clause for $\Box\varphi$, that $\models^T_{t_K} \Box\varphi$. Hence, if $\Box\varphi \in K$, then $\models^T_{t_K} \Box\varphi$.

For the converse direction, assume that $\models^T_{t_K} \Box\varphi$ and show that $\Box\varphi \in K$. Note that, by the truth clause for $\Box\varphi$, $\models^T_t \varphi$, for all $t \in T$, and hence, in particular, $\models^T_{t_K} \varphi$; therefore, by the inductive hypothesis, $\varphi \in K$. To show $\Box\varphi \in K$, suppose, by *reductio*, $\Box\varphi \notin K$. Then, because $K \in MC_{S5}$, $\neg\Box\varphi \in K$; and therefore, by (2) above, there is a $K' \in \Theta$ such that $\neg\varphi \in K'$. But then $\varphi \notin K'$, and, by the inductive hypothesis, $\not\models^T_{t_{K'}} \varphi$, which is impossible, because $\models^T_t \varphi$, for all $t \in T$, and $t_{K'} \in T$. Finally, by cases 1–4, it follows by the induction principle that $FM \subseteq \Delta$. $\blacksquare$

Theorem 46 *If K, $K' \in MC_{S5}$, $t_K = t_{K'}$, and for all $\varphi \in FM$, $\Box\varphi \in K$ iff $\Box\varphi \in K'$, then $K = K'$.*

Proof. Assume the hypothesis and let $\Delta = \{\Gamma \in MC_{S5} :$ for all φ, if $\Box\varphi \in K$, then $\varphi \in \Gamma\}$ and $T = \{t_\Gamma : \Gamma \in \Delta\}$. Then K, $K' \in \Delta$, and therefore by condition (3) of theorem 45 above, we have for all $\varphi \in FM$, both $(\varphi \in K$ iff $\models^T_{t_K} \varphi)$ and $(\varphi \in K'$ iff $\models^T_{t_{K'}} \varphi)$; and hence, because $t_K = t_{K'}$, $\varphi \in K$ iff $\varphi \in K'$, from which it follows that $K = K'$. $\blacksquare$

Corollary 47 *If K, $K' \in MC_{S5}$, $t_K = t_{K'}$, and for all φ, $\Diamond\varphi \in K$ iff $\Diamond\varphi \in K'$, then $K = K'$.*

Proof. By preceding theorem and definition of $\Diamond$. $\blacksquare$

Theorem 48 *If Γ v-implies φ, then $\Gamma \vdash_{S5} \varphi$.*

Proof. Assume the hypothesis. By the $S5$-analogue of theorem 29 (the proof of which is essentially the same as for L_{at}), it suffices to show that for all $K \in MC_{S5}$, if $\Gamma \subseteq K$, then $\varphi \in K$. Assume, accordingly, that $K \in MC_{S5}$ and that $\Gamma \subseteq K$. Let $\Delta = \{K' \in MC_{S5} :$ for all ψ, if $\Box\psi \in K$, then $\psi \in K'\}$ and $T = \{t_K : K \in \Delta\}$. Then $K \in \Delta$, and by condition (3) of theorem 45, for all $\psi \in K$ iff $\models^T_{t_K} \psi$. But $\Gamma \subseteq K$; therefore, $\models^T_{t_K} \psi$, for all $\psi \in \Gamma$. By the hypothesis, then, it follows that $\models^T_{t_K} \varphi$; and therefore $\varphi \in K$. $\blacksquare$

By theorems 44 and 49 together, we have our strong completeness theorem, from which the weak completeness follows as a corollary.

Theorem 49 *(Strong Completeness)*: *K v-implies φ iff $K \vdash_{S5} \varphi$.*

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Corollary 50 *(Weak Completeness):* φ *is valid iff* $\vdash_{S5} \varphi$.

Just as there is a secondary notion of validity corresponding to the primary notion of L-truth, and a secondary notion of v(alid)-implication corresponding to L-implication, so too we have a secondary notion of semantic consistency with respect to which we another version of the strong completeness theorem for $S5$.

Definition 51 Γ *is semantically consistent$_2$ iff for some $T \subseteq V$ and for some $t \in T$, for all $\varphi \in \Gamma$, $\models_t^T \varphi$.*

Theorem 52 Γ *is semantically consistent$_2$ iff Γ is $S5$-consistent.*

Proof. Essentially the same as for theorem 37. ■

Finally, let us prove here what we claimed earlier, namely that with respect to the "cut down" semantics for $S5$, there are possible worlds (as represented by maximally $S5$-consistent sets) that have the same atomic facts and yet that are not identical.

Lemma 53 *There are $K, K' \in MC_{S5}$ such that $t_K = t_{K'}$ and yet $K \neq K'$.*

Proof. Let $\Gamma = \{P_n : n \in \omega\} \cup \{\Box P_1\}$, and let $t(P_n) = 1$ for all $n \in \omega$. Also, let $\Gamma' = \{P_n : n \in \omega\} \cup \{\neg\Box P_1\}$ and let

$$t'(P_n) = \left\{ \begin{array}{l} 1 \text{ for all } n \neq 1 \\ 0 \text{ for } n = 1 \end{array} \right. ;$$

and finally let $T = \{t\}$ and $T' = \{t, t'\}$. Then for all $\varphi \in \Gamma$, $\models_t^T \varphi$ and for all $\varphi \in \Gamma'$, $\models_{t'}^{T'} \varphi$; i.e., Γ, Γ' are semantically consistent$_2$. Therefore, by theorem 53 above, Γ and Γ' are $S5$-consistent, and hence, by Lindenbaum's lemma, there are $K, K' \in MC_{S5}$ such that $\Gamma \subseteq K$ and $\Gamma' \subseteq K'$. But clearly because $\Box P_1 \in K$ and $\neg\Box P_1 \in K'$, $K \neq K'$, and yet, because $\Gamma \cap \Gamma' = \{P_n : n \in \omega\}$, we have it that $t_K = t = t_{K'}$. ■

8   Concluding remarks

Carnap's criterion of adequacy for logical necessity leads, as we have seen, to an intuitively natural semantics for logical necessity, even if only with respect to the metaphysical framework of logical atomism (which may be the only metaphysical framework in which a coherent account of logical necessity as the modal counterpart of logical truth can be given). This kind of semantics for logical necessity has been given for modal predicate logic as well, but with mixed results in regard to the question of completeness.[11] The kind of results we established here for propositional modal logic can be shown for modal *monadic* predicate logic as well, which, incidentally, like modal-free monadic predicate logic is also

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[11] See Cocchiarella 1975, and sections 1-2 of Cocchiarella 1984.

decidable. But once relational predicates and infinite domains are brought into the picture, then, instead of a completeness theorem, what can be shown is an essential incompleteness theorem. Such a result does not affect the philosophical significance of the semantics of logical necessity, but only the possibility of a complete (recursive) axiomatization for the full modal-predicate logic of logical necessity, and in that respect its philosophical significance is no less impaired than is that of arithmetic which is also incomplete with respect to its intuitively natural semantics.

The philosophical significance of the "cut-down" semantics of the restricted notion of necessity is another matter altogether. The semantics is philosophically defective in at least one respect: namely, that no explanation or rationale is given for the restricted interpretation of 'all possible worlds' in the semantical clause for necessity. This is not to say that such a rationale cannot be given (with respect, e.g., to a temporal or causal framework). To be sure, a "cut-down" of the notion of all possible worlds does provide the basis for a secondary notion of validity, and in particular a notion of validity with respect to which $S5$ is complete.[12] But such a result cannot alone be the grounds for accepting such a secondary notion of all possible worlds. What is needed is an independent semantical principle that provides a conceptual ground for such a "cut-down" of the meaning of 'all possible worlds' in the semantics for $\square$.[13]

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[12] But, as shown in Kripke 1962, modal monadic predicate logic is not decidable when interpreted with respect to this "cut-down" version of necessity. Indeed, unlike the situation in modal-free monadic predicate logic, relational content can be expressed in terms of monadic predicates and modal operators. (See Cocchiarella 1984, section 3, for a discussion of this issue.)

[13] For a discussion and an account of several such semantical principles see Cocchiarella 1984, especially section 15. Simply calling such a notion of necessity "metaphysical", incidentally, does not amount to providing such an account.

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